Bad Company Tamed

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Abstract

The neo-Fregean project of basing mathematics on abstraction principles faces "the bad company problem," namely that a great variety of unacceptable abstraction principles are mixed in among the acceptable ones. In this paper I propose a new solution to the problem, based on the idea that individuation must take the form of a well-founded process. A surprising aspect of this solution is that every form of abstraction on concepts is permissible and that paradox is instead avoided by restricting what concepts there are.

One of the most serious problems facing the neo-Fregean project of basing mathematics on abstraction principles is the so-called "the bad company problem." The problem is that a great variety of unacceptable abstraction principles are mixed in among the acceptable ones. A classic example of such a "bad companion" is Frege's inconsistent Basic Law V, which is logically very similar to the neo-Fregeans' favorite abstraction principle, namely the consistent and philosophically attractive Hume's Principle. The bad company problem shows that a deeper understanding is needed of the conditions under which abstraction is permissible.

The aim of this paper is to explore a new attempt to provide such an understanding, based on the idea that individuation must take the form of a well-founded process. Since an abstraction principle is naturally regarded as a device for individuating one class of entities in terms of another, this idea can be used to motivate restrictions on our theory of abstraction. In the first and less technical half of the paper (Sections 1 - 6) I develop the idea of individuation as a well-founded process in a modal framework. A surprising consequence of this development is a maximally permissive line on abstraction, according to which *any* form of abstraction on *any* concept is permitted and where paradox is instead avoided by restricting what concepts there are. In the second and more technical half (Sections 7 - 9) I articulate a natural and plausible restriction on the individuation of concepts. I prove that this restriction creates a safe environment for abstraction, where what used to be the bad companions now re-emerge as perfectly good ones.

1 The bad company problem

According to Frege, the fundamental law governing the identity of natural numbers is *Hume's Principle*, which says that the number of Fs is identical to the number of Gs if and only if the Fs and the Gs can be one-to-one correlated. Let $F \approx G$ be the pure second-order statement that there is a relation R that one-to-one correlates the Fs and the Gs. This defines an equivalence relation on concepts. Hume's Principle can then be formalized as:

$$\text{(HP)} \qquad \#F = \#G \;\leftrightarrow\; F \approx G$$

This principle has two highly desirable properties: It is consistent with full second-order logic; and it suffices, along with some very natural definitions, to derive all the axioms of Dedekind-Peano arithmetic.[1]

Encouraged by this, neo-Fregeans are interested in the broader class of principles to which (HP) belongs, namely principles of the form

$$(*) \qquad \S\alpha = \S\beta \leftrightarrow \alpha \sim \beta$$

where α and β are variables of some sort, and where $\sim$ is an equivalence relation on entities of this sort. Such principles are called *abstraction principles*, and $\S\alpha$ is said to be *the abstract* of α with respect to the equivalence relation $\sim$.

Unfortunately, not all abstraction principles are acceptable. The most famous example of an unacceptable abstraction principle is Frege's Basic Law V, which purports to describe how extensions are abstracted from concepts:

$$\text{(V)} \qquad \hat{F} = \hat{G} \;\leftrightarrow\; \forall x(Fx \leftrightarrow Gx)$$

As is well know, this gives rise to Russell's paradox. But other unacceptable abstraction principles are even more sinister because of their similarity to the acceptable ones. For

[1] For nice expositions of these results, see respectively [Boolos, 1987] and [Boolos, 1990]. For an overview of the neo-Fregean project and the bad company problem which is more extensive than that provided in this section, see [Linnebo, 2008].

instance, since equinumerosity of concepts is just a matter the concepts' being isomorphic, (HP) can be read as saying that two numbers are identical if and only if they are associated with isomorphic concepts. Consider now the abstraction principle which does to dyadic relations what (HP) does to concepts, that is, the abstraction principle which says that the *isomorphism types* of two dyadic relations are identical if and only if the relations are isomorphic:

(H^{2}P) $$\dagger R = \dagger S \leftrightarrow R \simeq S$$

This principle seems just as innocent Hume's Principle. Nevertheless it turns out to be inconsistent, as it allows us to reproduce the Burali-Forti paradox. A great variety of other "bad companions" are known as well.

What all of these "bad companions" show is that abstraction is extremely risky. Attractive and seemingly acceptable abstraction principles are surrounded by unacceptable ones, which they often closely resemble. This is not yet *an objection* to the abstractionist programme. After all, history is full of risky projects that have nevertheless succeeded. But the bad company problem highlights the need for an account of what kinds of abstraction are legitimate. We would like to draw a mathematically informative and philosophically well-motivated line between the acceptable abstraction principles and the unacceptable ones. This explanatory demand seems perfectly reasonable. For it is part of the very nature of both philosophy and mathematics to seek general principles and explanations wherever such are possible.[2]

Several attempts have been made to meet this explanatory demand.[3] The most influential attempt is based on the idea that an abstraction principle is acceptable just in case it is *stable*, where an abstraction principle ($*$) is said to be stable if there is a cardinal number κ such that ($*$) is satisfiable in all domains of cardinality greater than κ. But this approach faces a number of difficulties. Firstly, there is a worry about its ability to deal with *systems* of abstraction principles, as opposed to individual such principles.[4] Secondly, Gabriel Uzquiano's contribution to this special issue shows that this approach is incompatible with mathematical theories (such as ZFC set theory) that are presumed to be in good standing. Thirdly, Matti Eklund's contribution argues that this approach is poorly motivated philosophically.

Another approach to the explanatory demand generated by the bad company problem

[2]As Frege would be the first to admit: see [Frege, 1953], §§1-2.
[3]See [Linnebo, 2008], Section 3 for more details and references.
[4]See [Linnebo and Uzquiano,].

is very different. Where the standard approach just mentioned seeks to restore consistency by deeming various abstraction principles unacceptable, this alternative approach regards all abstraction principles, including Basic Law V, as acceptable and instead restores consistency by weakening the background second-order logic. More precisely, this approach proposes that we restore the second-order comprehension scheme

(Comp) $$\exists R \forall x_1 \ldots \forall x_n [Rx_1, \ldots, x_n \leftrightarrow \phi(x_1, \ldots, x_n)]$$

to *predicative* instances, that is, to instances where ϕ does not contain any bound second-order variables. The resulting theory is known to be consistent.[5] But since its logical strength is severely limited, this approach—at least as it stands—is not very appealing.[6]

2 Individuation as a well-founded process

The question emerging from the previous section is whether it is possible to draw a mathematically informative and philosophically well-motivated line between the acceptable abstraction principles and the unacceptable ones. The goal of this paper is to develop a new approach to this question, based on the idea that individuation must take the form of a well-founded process. I begin in this section by describing a framework in which this idea can be developed.

We may think of the process of individuation as one by which the ontology of mathematics is introduced.[7] This process is subject to the following five *Principles of Individuation*.

- Entities are introduced successively through a well-ordered series of stages.
- The introduction of an entity consists in the specification of its identity condition.

For instance, since the identity condition of a set is a matter of its elements, it suffices for the introduction of a set to specify what elements it has. And since the identity condition of a concept is a matter of its condition of application, it suffices for the introduction of a concept to specify such a condition.[8]

[5] See [Heck, 1996].

[6] See [Burgess, 2005], Sections 2.5 and 2.6.

[7] Related ideas are found in Kit Fine's "procedural postulationism"; see e.g. [Fine, 2002] and [Fine, 2005b]. Also related is Frege's infamous "proof of referentiality" in §§29-31 of [Frege, 1964]; for discussion and analysis, see [Linnebo, 2004].

[8] Presumably, in order to introduce a contingently existing object, more would have to be done than merely to stipulate identity conditions. But this complication need not detain us here, as our present concern is with mathematical objects, which exist necessarily if at all.

- Every entity maintains its identity condition throughout the rest of the process.

This is a reasonable requirement, which records the widely accepted belief that the identity condition of an entity is essential to it.

- *Groundedness.* The identity condition of an entity E may only presuppose entities individuated before E.

As it stands, this important principle is obviously rather vague. A way of making it precise will be proposed in Section 7.

- *Future Freedom.* No entity may be individuated which restricts the future freedom to individuate new entities in accordance with the Principles of Individuation.

This principle ensures that whenever we have a choice which entity to individuate, the order in which we choose to proceed is irrelevant. No generality is therefore lost by assuming that the series of stages is linear.

I will call each stage of the process of individuation a *coherent world.* The ontology of a coherent world consists of the objects and concepts individuated thus far. The coherent world also specifies how these entities are related and in this way determines an answer to every question about these entities that is expressible in the relevant language. A coherent world is thus much like a possible world. The only difference is that a coherent world need not contain every mathematical object, whereas a possible world must do so, given that mathematical objects are necessary existents.[9]

We next define an *accessibility relation* on coherent worlds. Say that a coherent world w *accesses* another such world w' (in symbols: wRw') just in case w' is a (not necessarily proper) extension of w; that is, iff w' contains all the entities of w and possibly more. The accessibility relation R will thus be reflexive, anti-symmetric, and transitive. Next we observe that the Principle of Future Freedom requires that the relation R be *directed*; that is, that R be such that whenever uRv and uRv' there is some w such that vRw and $v'Rw$. For unless R was directed, our choice whether to extend the ontology of u to that of v or that of v' would have a lasting effect, which would violate the Principle of Future Freedom. Finally, the requirement that individuation be well-founded corresponds to the requirement that R be well-founded.

[9] Another option would be to develop an explicit theory of stages, which quantifies over the stages themselves. My reasons for choosing the modal framework are purely pragmatic.

The accessibility relation R allows us to regard a system of coherent worlds as a Kripke-model of a modal logic. Because of the mentioned properties of R, the relevant models will validate (some higher-order version of) the modal logic S4.2, where S4.2 is the system of modal propositional logic that arises from adding to some axiomatization of S4 the principle

$$\text{(G)} \qquad \Diamond\Box p \to \Box\Diamond p.$$

Moreover, since the domains always increase along the accessibility relations, these models will also validate the Converse Barcan Formula:

$$\text{(CBF)} \qquad \Box\forall x\phi(x) \to \forall x\Box\phi(x)$$

Our language $\mathcal{L}$ will be one of third-order modal logic.[10] There will thus be variables for first-level relations (that is, relations whose relata are objects) of various adicity (where the monadic ones will be referred to as *concepts*), and for second-level relations (that is, relations whose relata are first-level relations) of various adicity. There will thus be infinitely many different types. The type of a variable will typically be indicated simply by letting lower-case variables range over objects, upper-case variables range over first-level concepts or relations, and bold-face variables range over second-level concepts or relations. The language contains two predicates. First there is the identity predicate, which can be flanked by terms of any (equal) type; for instance, '$F = G$' is thus well-formed. Then there is the non-logical predicate '$\textsc{Abst}_{\mathbf{R}}(F, x)$' expressing that the object x is the abstract of the concept F with respect to the equivalence relation $\mathbf{R}$.[11]

This approach to the bad company problem is philosophically very attractive. It draws on and generalizes ideas that have proved to be successful in set theory and in semantics. According to the prevailing iterative conception, sets are individuated by their elements, and this individuation (or "set formation") is required to be well-founded. From semantics we are familiar with the idea that a definition has to be well-founded in order to be informative. But the notion of individuation can be regarded as a metaphysical analogue of the semantic notion of definition. It is thus natural to think that the process of individuation too needs to

[10]Since I will only be concerned with Henkin models, this language can equally well be regarded as one of many-sorted *first-order* modal logic.

[11]Subscripts could, if desired, be introduced to distinguish between identity predicates for different types. We could also, if desired, make do with a *second-order* modal language whose sole non-logical predicate is '$\textsc{Abst}_{\rho}(F, x)$', where ρ is a formula describing an equivalence relation on first-level concepts.

be well-founded in order to be informative. Although this philosophical picture of the notion of individuation obviously needs to be spelled out, these considerations lend plausibility to my guiding idea that the process of individuation must be well-founded.

3 Relating ordinary and modal theories of abstraction

Ordinary theories of abstraction are not concerned with the processes by which entities are individuated. Rather, in these theories we quantify over all objects and all relations regardless of the stage at which they are individuated. This means that the quantifiers of an ordinary non-modal theory of abstraction correspond to the complex strings $\Box\forall x$ and $\Diamond\exists x$ of my modal theory, and likewise for the higher-order quantifiers. I will refer to such strings as *modalized quantifiers*. The ordinary quantifiers of a non-modal theory of abstraction are thus in my modal framework analyzed into two components: a modal operator interpreted as described in the previous section, and a quantifier ranging over entities of the appropriate kind in the relevant coherent world. Other expressions, such as the connectives and the identity predicate, have the same meaning in both kinds of theories.

A modal theory of abstraction is thus capable of making finer distinctions than the non-modal theories. In this section I ensure that the additional distinctions available in the modal framework are fully compatible with the principles of the non-modal framework. Let me be more precise. When we translate a formula of a non-modal theory of abstraction in accordance with the above remarks, we arrive at a formula all of whose quantifiers are modalized. I will say that such formulas are *fully modalized*.[12] What I do in this section is to ensure that modal distinctions are irrelevant to fully modalized formulas. This ensures that the modal translations of ordinary non-modal formulas behave in the ordinary way.

Say that a condition $\phi(u_1, \ldots, u_n)$, all of whose free variables are displayed, is *stable* iff, whenever it holds (or fails to hold) of a sequence of objects at some coherent world, it continues to hold (or fail to hold) of that sequence of objects at all larger coherent worlds. More precisely, a condition $\phi(u_1, \ldots, u_n)$ is stable iff it satisfies the following two requirements:

(Stab+) $$\Box\forall u_1 \ldots \forall u_n(\phi(u_1, \ldots, u_n) \to \Box\phi(u_1, \ldots, u_n))$$

(Stab−) $$\Box\forall u_1 \ldots \forall u_n(\neg\phi(u_1, \ldots, u_n) \to \Box\neg\phi(u_1, \ldots, u_n))$$

[12] Since it is a theorem of S4 that $\Box\forall x\forall y\, \phi$ and $\Diamond\exists x\exists y\, \phi$ are equivalent to respectively $\Box\forall x\Box\forall y\, \phi$ and $\Diamond\exists x\Diamond\exists y\, \phi$, I will also speak of formulas containing strings of the forms $\Box\forall x\forall y$ and $\Diamond\exists x\exists y$ as fully modalized.

(The case of $n = 0$ corresponds to there being no quantifiers.)

In an ordinary non-modal framework a predication has its truth-value absolutely, not just relative to some coherent world. In a modal framework this absoluteness corresponds to the assumption that all predicates are stable. We therefore make the customary assumption that the identity predicate is stable:

(Stab-Id+) $$\Box\forall x\forall y(x = y \to \Box x = y)$$

(Stab-Id−) $$\Box\forall x\forall y(x \neq y \to \Box x \neq y)$$

We also assume that our abstraction predicates are stable:

(Stab-Abst+) $$\Box\forall F\forall x(\textsc{Abst}_{\mathbf{R}}(F, x) \to \Box\textsc{Abst}_{\mathbf{R}}(F, x))$$

(Stab-Abst−) $$\Box\forall F\forall x(\neg\textsc{Abst}_{\mathbf{R}}(F, x) \to \Box\neg\textsc{Abst}_{\mathbf{R}}(F, x))$$

Finally we require that every relation R be stable:

(Stab-Pred+) $$\Box\forall x_1 \ldots \forall x_n(Rx_1 \ldots x_n \to \Box Rx_1 \ldots x_n)$$

(Stab-Pred−) $$\Box\forall x_1 \ldots \forall x_n(\neg Rx_1 \ldots x_n \to \Box\neg Rx_1 \ldots x_n)$$

I will refer to these three pairs of axioms as *the stability axioms*. Given the stability axioms, we get the following lemma.

Lemma 1 Every fully modalized $\mathcal{L}$-formula is stable.

Proof. We proceed by induction on the complexity of ϕ. If ϕ is atomic, then the claim follows directly from the stability axioms. If the principal operator of ϕ is a truth-functional connective, the induction step is trivial. Finally, assume ϕ is $\Box\forall x\psi(x)$ and that this formula holds at w. Then it continues to hold at any extension w' of w. The same goes for the other modalized quantifiers. $\dashv$

This lemma ensures that any relation definable by a fully modalized formulas is stable, in accordance with (Stab-Pred+) and (Stab-Pred−). By contrast, formulas containing non-modalized quantifiers tend to be unstable. Consider for instance the formula $\phi(x)$ which says that x is one of only four objects. This formula can hold of an object at one stage but cease to do so as soon as another object is individuated.

The next lemma shows that modal distinctions collapse in connection with stable formulas.

Lemma 2 Let ϕ be a stable formula. Then ϕ, $\Diamond\phi$, and $\Box\phi$ are semantically equivalent.

Proof. As usual, $\Box\phi \rightarrow \phi$ and $\phi \rightarrow \Diamond\phi$ are valid. So it suffices to show that $\Diamond\phi \rightarrow \Box\phi$ is valid as well. Assume $\Diamond\phi$ is true at some world w. Then by stability we must have ϕ already at w. But then by stability again, $\Box\phi$ is true at w. $\dashv$

Combining these two lemmas with the fact that a non-modal formula translates as a fully modalized formula, we get our desired result that modal distinctions are irrelevant to the translations of ordinary non-modal formulas into my modal framework. In particular, the fully modalized formulas that this translation results in may be assigned truth-values in an absolute way, not just relative to some coherent world.

This does not mean that modal distinctions are irrelevant to our investigation in general. On the contrary, the next sections will show that modal distinctions are invaluable when analyzing what is required for the successful individuation of various kinds of entity.

4 The individuation of concepts

We will now be concerned with the concept comprehension scheme

$$(X\text{-C}\exists) \qquad \Diamond\exists R\Box\forall x_1 \ldots \forall x_n[Rx_1 \ldots x_n \leftrightarrow \phi(x_1, \ldots, x_n)].$$

This scheme tells us that every open formula $\phi(x_1, \ldots, x_n)$ from some class X can be used to individuate a relation R which holds of actual or future objects $x_1, \ldots, x_n$ just in case $x_1, \ldots, x_n$ satisfy the formula $\phi(x_1, \ldots, x_n)$. Note that (X-C$\exists$) is just the modal translation of the ordinary non-modal concept comprehension scheme (Comp) from Section 1. Note also that the formula $\phi(x_1, \ldots, x_n)$ may contain free variables other than the x_i, including higher-order ones. Such variables are called *parameters* and are tacitly understood to be bound by modalized universal quantifiers whose scope is the entire formula shown in (X-C$\exists$). The parameters can thus be thought of just as terms denoting appropriate entities. I will refer to an open formula, possibly containing parameters, as a *condition*, and I will display only those of its variables that are not parameters, as in (X-C$\exists$) above. There will also be an analogous comprehension scheme for second-level concepts. For simplicity, I will in what follows only be concerned with the comprehension scheme for monadic first-level concepts. But my analysis of this special case can easily be extended to the the other cases.

What is required of a condition $\phi(x)$ for it to define a concept? A minimal requirement is that the condition $\phi(x)$ be stable. But stability alone does not guarantee that a condition is suitable to individuate a concept. (For instance, we will see in Section 6 that the condition needed to generate a version of Russell's paradox is fully modalized and therefore stable.) To identify what more is required, we need to examine how concepts are individuated.

The basic structure of concept individuation is quite simple. A concept is specified by means of some absolutely general condition of application, which is true of those objects that fall under the concept and false of those that do not. Moreover, two such conditions specify the same concept just in case satisfaction of one ensures satisfaction of the other. The hard question is how to make this rough idea precise. What is it for a condition to be "absolutely general" and for satisfaction of one condition to "ensure" satisfaction of another?

The standard Fregean view on the individuation of concepts is that two concepts are identical just in case they are coextensional.[13] But this view is stated using ordinary quantifiers, which correspond to the modalized quantifiers of my modal framework. When the Fregean view is translated into my modal framework, we thus get the following criterion of identity:

(C=) $$F = G \leftrightarrow \Box\forall x(Fx \leftrightarrow Gx)$$

This account of the identity condition for concepts suggests an analysis of the "absolute generality" that a condition must possess in order to specify a concept. For we now see that such a condition must be defined on absolutely all objects *including such as have yet to be individuated*. So for a condition $\phi(x)$ to define a concept at some world w, we need a guarantee that the condition will remain defined on any objects that may be individuated at later worlds. Precisely what this requirement amounts to depends on what abstracts we are allowed to individuate—of which more shortly.

This Fregean view of concept individuation not only has a noble pedigree but is very plausible in its own right. For it is in the nature of concepts to "reach out" ahead of the stage at which they are individuated and be capable of applying even to objects not yet individuated. The concept of an ordinal number provides a nice example of this absolute generality. Since this concept can be defined using only the membership predicate $\in$ and the resources of first-order logic, it will be defined on any set whatsoever, including such as

[13]Strictly speaking, Frege would not talk about identity holding between concepts. Rather, he would say that coextensionality is to concepts what identity is to objects. Henceforth I will ignore this complication.

have yet to be individuated. Moreover, if we regard the concept as trivially false of any non-set, the concept will be defined on absolutely every object, including such as have yet to be individuated. By contrast, assume that at some world we attempt to individuate a concept by providing a complete list of the objects *in that world* that fall under the concept. This attempt fails to individuate a concept, as we will not know which objects of later and larger worlds fall under it.

5 The individuation of abstracts

I now turn to the question how concepts and suitable equivalence relations can be used to individuate abstracts. Since we work in a third-order logic, the relevant equivalence relation can be represented by means of a two-place third-order relation $\mathbf{R}(F, G)$. As usual, this equivalence relation $\mathbf{R}$ is assumed to be stable.[14] Our task is then to formulate the identity conditions for abstracts.

So long as the two abstracts are associated with the same equivalence relation $\mathbf{R}$, the answer is straightforward:

$$\textsc{Abst}_{\mathbf{R}}(F, x) \wedge \textsc{Abst}_{\mathbf{R}}(G, y) \rightarrow [x = y \leftrightarrow \mathbf{R}(F, G)]$$

But what about the general case where x and y are abstracts associated with different equivalence relations? One plausible axiom for the identity of abstracts in general is the following:

$$(\text{A}{=}_1) \qquad \textsc{Abst}_{\mathbf{R}}(F, x) \wedge \textsc{Abst}_{\mathbf{S}}(G, y) \rightarrow [x = y \leftrightarrow \mathbf{R}{=}\mathbf{S} \wedge \mathbf{R}(F, G)]^{15}$$

Another plausible option is to identify the $\mathbf{R}$-abstract of F with the $\mathbf{S}$-abstract of G just in case two abstracts are associated with the same equivalence class of concepts, that is, just in case $\Box\forall H(\mathbf{R}(F, H) \leftrightarrow \mathbf{S}(G, H))$. But this latter formula can be simplified. Once the concepts $\mathbf{R}$, $\mathbf{S}$, F, and G have been individuated, we can also individuate the concepts $\lambda H.\mathbf{R}(F, H)$ and $\lambda H.\mathbf{S}(G, H)$. Using (C=) we can then rewrite the above formula as $\lambda H.\mathbf{R}(F, H) = \lambda H.\mathbf{S}(G, H)$. The second plausible axiom for the identity of abstracts can then be expressed

[14] If one preferred to remain within second-order logic, one could instead use stable *formulas* $\rho(F, G)$ with 'F' and 'G' free.

[15] Without third-order logic, the natural strategy would be to say that two equivalence relations ρ and σ of equal type are *equivalent* (abbreviated $\textsc{Eqv}(\rho, \sigma)$) iff $\Box\forall F \forall G[\rho(F, G) \leftrightarrow \sigma(F, G)]$ and then to lay down: $\textsc{Abst}_{\rho}(F, x) \wedge \textsc{Abst}_{\sigma}(G, y) \rightarrow [x = y \leftrightarrow \textsc{Eqv}(\rho, \sigma) \wedge \rho(F, G)]$.

as

$$\text{(A=}_2\text{)} \qquad \textsc{Abst}_{\mathbf{R}}(F,x) \wedge \textsc{Abst}_{\mathbf{S}}(G,y) \rightarrow [x = y \leftrightarrow \lambda H.\mathbf{R}(F,H) = \lambda H.\mathbf{S}(G,H)].$$

For present purposes we need not take a stand on which of these two identity conditions is more plausible.[16] Nothing in what follows turns on which condition we choose.

A stable equivalence relation $\mathbf{R}$ and a concept F in its field suffice to individuate an abstract. To see this, assume we have $\textsc{Abst}_{\mathbf{R}}(F,x)$ in some coherent world w, and consider any object y. If y is not in w, then we must have $x \neq y$. Likewise, if y is not an abstract, we must have $x \neq y$. So assume that y is in w and that it is the abstract of G with respect to some stable equivalence relation $\mathbf{S}$. Then (A=$_1$) (or (A=$_2$)) settles whether $x = y$.

6 In favor of unrestricted abstraction

Having explained how concepts and abstracts are individuated, I now turn to the important question what concepts and abstracts there are. Recall that for a condition $\phi(u)$ to define a concept at some coherent world w, the condition must remain defined on any objects that may be individuated at later worlds. There are two opposite ways of enforcing this requirement.[17]

The first way is to regard the requirement as a restriction on what *abstracts* can legitimately be individuated. No restrictions are imposed on the concept comprehension scheme, other than the minimal one of stability. Against this background assumption of unrestricted concept comprehension, only such abstracts are allowed to be individuated as preserve the fact that all concepts remain universally defined.

The second way to enforce the requirement is to regard it as a restriction on what *concepts* can legitimately be individuated. No restrictions are imposed on the introduction of abstracts, other than the minimal one of stability. This corresponds to adopting the axiom

$$\text{(A}\exists\text{)} \qquad \Box\forall\mathbf{R}\forall F\Diamond\exists x\, \textsc{Abst}_{\mathbf{R}}(F,x),$$

where $\mathbf{R}$ is tacitly understood to range over stable equivalence relations. Against this background assumption of unrestricted abstraction, only such concepts are allowed to be individuated as are guaranteed to remain defined on any abstracts we may go on to individuate.

[16] For discussion, see [Cook and Ebert, 2005] and [Fine, 2002], Section I.5.

[17] Hybrid approaches may be possible as well but will not be considered here. Frege's mistake in his "proof of referentiality" was to disregard the requirement entirely; see [Linnebo, 2004].

The approach to the bad company problem that is currently most influential—the one based on the notion of stability—falls into the former, comprehension-friendly category. Indeed, unrestricted concept comprehension is often regarded as an obvious and uncontroversial logical truth. Against this prevailing view I will now defend the alternative, abstraction-friendly approach. My argument will make essential use of the idea of individuation as a well-founded process and of the explication of this in terms of coherent worlds.

This framework brings out an important difference between the ways in which concepts and abstracts are individuated. The individuation of an abstract is inherently "backward-looking" in that it refers only to concepts and relations that have already been individuated. This kind of individuation is thus completely unproblematic as far as the Principle of Groundedness is concerned. By contrast, the individuation of a concept is inherently "forward-looking" in that (C=) uses modalized quantifiers that range over all objects, including such as will be individuated only after the concept in question. The individuation of a concept is therefore risky in a way that the individuation of an abstract is not: Its form is such that it is prone to presuppose entities that are individuated only after the concept in question, in violation of the Principle of Groundedness.

This argument will have to await the analysis of the crucial notion of presupposition—to be developed in the next section—for its final expression. Even so, a good sense of the problematic nature of concept individuation can be obtained by considering a version of Russell's paradox. Let $\textsc{Ppty}(F, x)$ be the statement that x is the property of being F, in the sense that x is the abstract of F with respect to the equivalence relation of concept identity. Consider the "Russell condition" $\rho(x)$ given by the formula $\Diamond\exists F[\textsc{Ppty}(F, x) \wedge \neg Fx]$. Assume this condition could be used to individuate a concept R, which would then exist at some coherent world w. We could then introduce the associated property r at some later world w', where we would then have:

$$Rr \leftrightarrow \Diamond\exists F[\textsc{Ppty}(F, r) \wedge \neg Fr]$$

But standard reasoning shows that this formula leads to contradiction.[18] What has gone wrong? Because of its modalized quantifier $\Diamond\exists F$, the Russell condition $\rho(x)$ is concerned

[18] Assume $\neg Rr$ in w'. Then $\Box\forall F(\textsc{Ppty}(F, r) \rightarrow Fr)$, against which R would itself be a counterexample. So we must have Rr in w'. This means there is some world w'' extending w' with a concept F such that $\textsc{Ppty}(F, r)$ and $\neg Fr$. Since in w'' both R and F have r as their property, R and F must be identical. But this conflicts with the fact that in w'' we have Rr (by stability) and $\neg Fr$.

with concepts not yet individuated, including the concept R that the condition purports to individuate. In fact, when the laws governing properties are taken into account, the condition specifies that the concept R is to apply to its own property r just in case it does not so apply. So by being concerned with entities not yet individuated, the Russell condition ends up making an inconsistent requirement on the behavior of the concept R.

Summing up, I have argued that the individuation of abstracts is unproblematic from the point of view of the Principle of Groundedness, whereas the individuation of concepts is highly problematic. In the remainder of this paper I will therefore be concerned with abstraction-friendly theories rather than the more widely investigated comprehension-friendly ones. My approach can thus be summed up in the following slogan: Ask not what abstraction is permitted given unrestricted comprehension; ask what comprehension is permitted given unrestricted abstraction.

7 The notion of determination

A systematic investigation of abstraction-friendly theories would be a huge undertaking. All I can do here is scratch the surface. My present aim will be the modest one of providing some examples of what such theories can look like and of the mathematics that they enable us to develop. One important task which will have to await another occasion is to show how my account of individuation as a well-founded process can be used to motivate the axioms of ZFC set theory. My present goal will be the more modest one of showing how this guiding idea is compatible with ordinary set theory.

Any abstraction-friendly theory will include the stability axioms, the abstraction axioms (A$\exists$) and (A$=_1$) (or (A$=_2$)), and the concept axioms (C$=$) and (X-C$\exists$) for some suitable class X of comprehension conditions. I will refer to this theory as X-AF. Our most pressing question concerns the class X: What is required of a comprehension condition for it to be suitable to define a concept? Restrictions on these conditions will most likely come from the Principle of Groundedness, which tells us that the individuation of a concept may only "presuppose" entities that are individuated before the concept in question. But what is it for the individuation of a concept to "presuppose" an entity? Recall that the identity of a concept is a matter of the distinction it draws between objects—those to which it applies and those to which it doesn't. The "presuppositions" involved in the individuation of a concept should therefore be a matter of what entities are relied upon in drawing this distinction.

In order to make these intuitive considerations precise, I will now adopt what Stewart Shapiro in [**?**] calls *the external perspective*, namely the perspective of a bystander who is interested in theories of abstraction against the background of standard mathematics. Unless otherwise stated I will be working in a meta-theory consisting of ZFC set theory. This will allow me to give a precise mathematical definition of the intuitive notion of determination invoked above.[19] Very roughly, a condition will be said to be *determined* by some entities just in case the individuation of these entities suffices to fix the behavior of the condition on the relevant domain, thus ensuring that the behavior of the condition remains unaffected by what entities we go on to individuate. The Principle of Groundedness can then be expressed as the requirement that for a condition ϕ to individuate a concept, ϕ must be determined by entities already individuated. My investigation of the notion of determination will show how to construct set-theoretic models for abstraction-friendly theories.

Since we are only concerned with fully modalized conditions, we need to consider a domain of all "possible entities." For a modalized quantifier in effect ranges over the union of the domains of all coherent worlds, which can be considered as the domain of all possible entities. We will use a set D_τ to represent the possible entities of each type τ. (As before, the type of an entity will for the most part be indicated simply by using different styles of variable for entities of different type.) Let D be a set which for each type τ specifies a domain D_τ of such entities (for instance by containing, for each type τ, a unique ordered pair $\langle \tau, D_\tau \rangle$ whose first entry is τ). For reasons that will become clear shortly, it makes sense to require that all the domains D_τ be of the same (infinite) cardinality. I will call a domain D of this sort an *abstraction domain.*

Given an abstraction domain D, we need a way to represent what entities have been individuated, and how. I will do this by means of certain sets which I will call *individuation reports for D*. An individuation report I for D consists of entities from D and n-tuples of such entities. When I contains an entity from D, this represents that the entity has been individuated. When I contains a triple of the form $\langle \mathbf{R}, F, x \rangle$, this represents that the $\mathbf{R}$-abstract of F is x. When I contains an $n+1$-tuple whose first element is an n-adic concept of a type that is applicable to the next n elements (in that order), this represents that the concept applies to these elements (in that order). For instance, $\langle F, x \rangle \in I$ represents that the object x falls under the first-level monadic concept F, and $\langle \mathbf{X}, F \rangle \in I$ represents that the the first-level

[19] This definition is inspired by [Leitgeb, 2005].

concept F falls under the second-level concept $\mathbf{X}$. Finally, every individuation report I must meet the following two requirements. When I represents an entity as individuated, it must also specify *how* this entity is individuated. And when I represents x as the $\mathbf{R}$-abstract of F, I must also represent $\mathbf{R}$ and F as individuated: for an abstract can only be individuated by means of concepts that have themselves been individuated.

Let $I \sqsubseteq J$ ("I is extended by J") mean that every entity that I represents as individuated in some way is also represented by J as individuated in the same way. So in particular, $I \sqsubseteq J$ requires that every concept which I represents as applying to certain actual and possible entities be represented by J as applying to precisely the same entities.

Given an individuation report I for some abstraction domain D, we can interpret $\mathcal{L}$-formulas in the obvious way by letting an atomic formula be satisfied by a string of entities just in case I represents these entities as related in the appropriate way. For instance, we let '$\textsc{Abst}_{\mathbf{R}}(F, x)$' be satisfied by $\langle \mathbf{S}, G, y \rangle$ just in case I represents y as the $\mathbf{S}$-abstract of G. Let $[\![\phi(x)]\!]_I$ be the set of objects of D that satisfy the condition $\phi(x)$ when this condition is interpreted in accordance with I.

We can now finally give a precise definition of our target notion of determination. Say that a condition $\phi(x)$ is *determined on D by I* iff every individuation report J that extends I assigns to $\phi(x)$ the same extension as I does:

$$\forall J(I \sqsubseteq J \rightarrow [\![\phi(x)]\!]_I = [\![\phi(x)]\!]_J)$$

Satisfaction of this requirement means that the behavior of $\phi(x)$ on D is determined already by the entities that I represents as individuated. No matter what entities we go on to individuate, this won't affect the behavior of the condition $\phi(x)$ on entities from D. Conditions that are determined in this way are therefore suitable for individuating concepts, as recorded in the following lemma.

Lemma 3 Let I be an individuation report for some abstraction domain D, and let ϕ be a condition that is determined on D by I. Assume D contains a concept F whose type makes it appropriate to be individuated by ϕ and which I represents as not yet individuated. Then there is an individuation report J extending I which represents F as the concept individuated by ϕ, and which is such that any further extension K of J continues thus to represent F.

Proof. Assume the condition in question is $\phi(x)$. (The case of conditions of other types is analogous.) Let F be a monadic first-level concept from D which I represents as not yet individuated. Let J be as I except that J also represents F as applying to all and only the objects in $[\![\phi(x)]\!]_I$. Since $\phi(x)$ is determined on D by I, we have $[\![\phi(x)]\!]_I = [\![\phi(x)]\!]_K$ for any $K \sqsupseteq I$. This means that J has the desired properties. $\dashv$

The notion of determination can also be used to explain why the Russell condition $\rho(x)$ from last section fails to individuate a concept. Russell's paradox is thus avoided in a systematic and independently motivated way, not by *ad hoc* restrictions.[20]

Lemma 4 Let I be an individuation report for some abstraction domain D which represents at least one object and at least one monadic first-level concept as not yet individuated. Then $\rho(x)$ is not determined on D by I.

Proof. Let a and F be respectively an object and a monadic concept from D which I represents as not yet individuated. Choose any individuation of the concept F, and then individuate a as its associated property. The condition $\rho(x)$ will then be satisfied by a just in case $\neg Fa$. But when we individuated F, we had a choice whether or not to make F apply to a. So the behavior of $\rho(x)$ on a cannot have been determined already by I. $\dashv$

I now turn to the task of constructing models of abstraction-friendly theories. The first step will be a generalization of Lemma 3. But first we need a definition. Consider an individuation report I on an abstraction domain D of some infinite cardinality κ. Say that I is *frugal* iff for each type τ, I represents κ entities of type τ as not yet individuated.

Lemma 5 Let I be a frugal individuation report on an abstraction domain D of some infinite cardinality κ. Then there is a frugal individuation report J extending I which represents all the conditions that are determined on D by I as individuating concepts, and all the equivalence relations and concepts that are then available as individuating abstracts. Although we have some choice in how to carry out these individuations, our choice doesn't matter, as the resulting individuation reports are all pairwise isomorphic.

Proof. The proof is for the most part an obvious generalization of that of Lemma 3. The only new challenge concerns the number of entities of D represented as not yet individuated.

[20] Other "bad companions" (in the sense of Section 1) are blocked in similar ways. For instance, the version of the Burali-Forti paradox that threatens (H^2P) is blocked because the formula $\psi(x, y)$ needed to define a well-ordering of the ordinals is subject to a lemma which says of it what Lemma 4 says of $\rho(x)$.

I claim there are at most κ many conditions determined by I on D. To see this, observe that there are countably many open formulas, each with finitely many parameters that can take on at most κ different values. This yields at most $\omega \cdot \kappa^{<\omega} = \kappa$ many such conditions. Our use of determined conditions to individuate concepts can thus be carried out so as to ensure that there are still κ "unused" concepts of each type τ. An analogous argument applies to the abstracts. The claim about isomorphism is obvious. $\dashv$

By the Axiom of Choice there is a function Γ that maps a frugal individuation report I to one of the individuation reports J resulting from Lemma 5. Let I_0 be some initial frugal individuation report. We will now construct a sequence of individuation reports based on the following rules:

- $I_{\alpha+1} = \Gamma(I_\alpha)$
- $I_\lambda = \bigcup_{\gamma<\lambda} I_\gamma$ for each limit ordinal λ of cardinality no greater than κ

The rule for successor ordinals is licensed by Lemma 5. And some simple cardinal arithmetic shows that Γ can be chosen so as to ensure that each I_λ remains frugal, provided $|\lambda| \leq \kappa$. (If, on the other hand, $|\lambda| > \kappa$ and there is no $\alpha < \lambda$ such that I_α is a fixed point of Γ, then the construction of I_λ cannot be carried out, as we would run out of new entities of D to individuate.)

Although the initial individuation report I_0 is required to be frugal, it is highly significant that it is not required to be empty. For this allows us to work relative to a class of entities of whose individuation we have an independent account. Of particular importance to us is the ability to work relative to an independently given conception of sets, such as the iterative conception. Assume for instance I_0 represents some of the objects of D as being borne a relation $\in$ by other objects of D. We may also assume the relation $\in$ to satisfy various axioms of set theory. Call the objects in the co-domain of $\in$ *sets*. We can then let an individuation report I represent a set x as individuated just in case I represents every element of x as individuated.

Theorem 1 Let I_0 be a frugal individuation report on an infinite abstraction domain D, and let λ be a limit ordinal such that $|\lambda| \leq |D|$. Then there is a sequence $\{I_\gamma\}_{\gamma\leq\lambda}$ of individuation reports, determined in accordance with the above rules, such that I_λ represents every entity of D as individuated. Associated with this sequence is a Kripke model of the theory $\varnothing$-AF.

Proof. With one exception, the construction of the sequence $\{I_\gamma\}_{\gamma\leq\lambda}$ is described above. The exception concerns how to ensure that I_λ represents every entity of D as individuated. If Γ has a fixed point $\alpha < \lambda$, then we choose I_α such that it represents every entity of D as individuated. If Γ has no such fixed point, then make an analogous choice concerning I_λ.

Next we observe that the sequence $\{I_\gamma\}_{\gamma\leq\lambda}$ is associated with a Kripke model based on the set of worlds $\{w_\gamma : \gamma \leq \lambda\}$ and an accessibility relation R such that $w_\gamma R w_\delta$ iff $\gamma \leq \delta$. Let the ontology of each world w_γ be that represented as individuated by I_γ, and let the truths concerning this ontology be precisely those represented by I_γ. I claim that all the axioms of $\varnothing$-AF come out true in this model. The stability axioms come out true because our definition of $\sqsubseteq$ required that when an individuation report I represents certain entities as individuated in a particular way, then any extended individuation report J must represent these entities as individuated in the same way. The axioms concerned with identity, (C=) and (A$=_1$) (or, if we prefer, (A$=_2$)) come out true because every individuation is carried out so as to validate these axioms. What remains is (A∃). Assume I_λ represents the equivalence relation $\mathbf{R}$ and the concept F as individuated. Then both are individuated under some I_α for $\alpha < \lambda$. Then the $\mathbf{R}$-abstract of F is individuated under $I_{\alpha+1}$. ⊣

Clearly, the Kripke model described in this theorem validates a variety of concept comprehension axioms as well. But since the notion of determination cannot be (directly) expressed in our object language, the task of identifying these comprehension axioms is non-trivial.

We will be particularly interested in comprehension axioms that are validated in a robust and systematic way. Say that a condition ϕ is *grounded on D by its parameters* iff ϕ is determined on D by every individuation report on D which represents all the parameters of ϕ as individuated.[21] Say that a condition ϕ is *grounded by its parameters* iff ϕ is grounded by its parameters on every infinite abstraction domain D. This latter kind of condition can plausibly be taken to satisfy the Principle of Groundedness and thus be appropriate for individuating concepts, not just in the above set theoretic models, but in a theory with a primitive notion of abstraction.

[21]The notion of a parameter was explained in Section 4.

8 Predicativity as a route to groundedness

In this section I identify a large and interesting class of conditions that are grounded by their parameters. The common characteristic of these conditions is a kind of predicativity. When motivating and explaining this class, it is useful to regard a condition as an idealized algorithm. We can then ensure that a condition is grounded by ensuring that the associated algorithm never draws on resources that are not yet available.

Let's explore this idea by considering two simple examples. Consider first the condition '$x \neq a$'. I claim that this condition is grounded by its single parameter a. Assume a belongs to an abstraction domain D, on which an individuation report I represents a as individuated. Given an object x, check whether x is identical to a. If it is, the condition does not apply. If it is not, the condition applies. Clearly, the only entity that this algorithm draws on is a. The behavior of the algorithm is thus determined on any abstraction domain D once the object a has been identified. The associated condition is therefore grounded by its parameter.

Consider next the condition '$\Diamond\exists y\Box\forall z(Rxz \leftrightarrow y = z)$'. I claim that this condition is grounded by its single parameter R. Assume R belongs to an abstraction domain D, on which an individuation report I represents R as individuated. Given an object x, check whether x bears R to some unique (actual or possible) object. (This task is possible because R, being individuated, is defined on all actual and possible objects of D.) If it is, the condition applies. If it is not, the condition does not apply. Clearly, the only entity that the algorithm draws on is R. The behavior of this algorithm is thus determined on D once the relation R has been identified. The associated condition is therefore grounded by its parameter.

Let's now attempt to draw some broader lessons from these examples. What ensures that the above two conditions are grounded by their parameters is that they contain no vocabulary that is sensitive to what entities we go on to individuate. The connectives and the identity predicate are not sensitive to such matters. (We may also introduce other predicates whose interpretation is fixed and thus not sensitive to such matters.) Nor are the modalized quantifiers, as they always range over the same possible entities (although more and more of these are regarded as actual). But the predicate 'ABST' *is* sensitive to what entities we go on to individuate, as it will apply to a new triple of entities every time a new abstract is individuated. The same goes for atomic subformulas of predication (that is, formulas of the form $Rx_1 \ldots x_n$). For every time a new relation is individuated, such formulas may be satisfied by new tuples of entities. But as the second example above shows, atomic subformulas of predication don't always introduce such sensitivity. So we need to distinguish between two kinds of occurrences of variables. Say that a variable *occurs in a demanding position* in a formula ϕ iff the variable occurs in predicate position of some atomic subformula of ϕ or it occurs in an atomic subformula of ϕ involving the predicate 'ABST'. So when a variable does *not* occur in a demanding position in ϕ, every occurrence of the variable in an atomic subformula of ϕ is in an argument position of some relation term. But as we have seen, when a relation is individuated it is defined on the entire domain, including entities not yet individuated. It follows that when a variable occurs only in non-demanding positions, then its value need not be required to be individuated.

Say that a condition ϕ is *non-demanding* when all of its variables that occur in demanding positions are used as parameters (rather than as bound variables or variables that we abstract on). What I have argued is thus that a non-demanding condition, regarded as an idealized algorithm, draws only on resources that are already available. This is reflected in the following theorem.

Theorem 2 Every non-demanding condition is grounded by its parameters.

Proof. We proceed by induction on ϕ. Any atomic subformula of ϕ is grounded by its parameters, as this subformula must either be based on the identity predicate, be of the form '$Rx_1 \ldots x_n$' where R is a parameter, or consist of the predicate 'ABST' applied to three parameters. As for the induction step, the only non-trivial case is that of the modalized quantifiers. So assume that the subformula ψ is grounded by its parameters and consider the formula $\Box\forall x\, \psi$. By assumption, every occurrence of the variable x in ψ is in a non-demanding position. This ensures that $\Box\forall x\, \psi$ too is grounded by its parameters. $\dashv$

Given Theorems 1 and 2 it is easy to construct models of abstraction-friendly theories with comprehension axioms for all non-demanding conditions. Consider the simplest such theory, namely the theory *ND-AF* in our basic language $\mathcal{L}$ with comprehension axioms for all non-demanding conditions.

Corollary 1 ND-AF is true in any model of the sort constructed in Theorem 1.

Proof. Consider the Kripke model associated with an abstraction domain D and an individuation report I_λ, as in the proof of Theorem 1. Given Theorem 1, all we need to do is verify that the comprehension axioms are true in this model. So consider a non-demanding condition ϕ. Since ϕ has only finitely many parameters, there must be some stage $\alpha < \lambda$ such that I_α represents all of ϕ's parameters as individuated. By Theorem 2, ϕ is grounded. By the definition of groundedness, ϕ is thus determined on D by I_α. But then $I_{\alpha+1}$ (and thus also I_λ) represents ϕ as individuating a concept. $\dashv$

A definition is ordinarily said to be *predicative* if it doesn't quantify over any totality to which the referent of the definiendum (if any) belongs. Although the class of non-demanding conditions fail to be predicative by the letter of this ordinary definition, they are still predicative by its spirit. For these conditions don't *in any demanding sense* quantify over entities not yet individuated. I will therefore regard non-demanding conditions as predicative. But a condition need not be non-demanding to be predicative. Another way to ensure that a condition is predicative is by restricting all of its variables that are not parameters but nevertheless occur in demanding positions to entities already individuated. Let me be more precise. Assume we can formulate in the object language a condition '$\textsc{Old}(x)$' which, given any individuation report I on any abstraction domain D is true of an object x only if x is an object already individuated under I. Assume we can formulate analogous conditions for other types of entities. Using these conditions we can require that every variable that is not a parameter but nevertheless occurs in a demanding position has as its value some entity that has already been individuated. Say that a condition ϕ is *predicative* just in case all of ϕ's variables that are not parameters but nevertheless occur in demanding positions are restricted in this way to entities already individuated. We then get the following theorem, which I state without proof.

Theorem 3 Every predicative condition is grounded by its parameters.

This theorem opens the way to some strong theories with interesting foundational properties. An example is provided in an appendix.

9 An application to cardinal numbers

I will end by outlining an application of my account of individuation as a well-founded process to the theory of cardinal numbers.

According to the standard neo-Fregean account, every concept F has a cardinal number $\#F$. This account is criticized by [Boolos, 1997], who objects that we have no reason to accept such things as the number of all objects (a number Boolos calls "anti-zero") and the number of all ordinals. The account developed in this paper allows us to give precise content to Boolos's objection and to formulate an attractive response. Recall that I have required that the assignment of any abstract be *stable*, in the sense that once an abstract is assigned to a concept, this abstract remain assigned to the concept no matter what entities we go on to individuate. Assume that a concept F is assigned a cardinal number κ at some world w. By stability it follows that κ must remain assigned to F at any larger world w'. But for a concept to be assigned the same cardinal number at two different worlds, the concept must have precisely as many instances at each of the two worlds—otherwise we would simply not be talking about cardinal numbers but about some other kind of abstract. This means that we cannot assign a cardinal number to a concept at any world at which it is possible to go on and individuate further instances of the concept. Since we can always go on and individuate further instances of the concept of being self-identical and the concept of being an ordinal, it follows that there is no world at which we can assign a cardinal number to these concepts.

In response to Boolos we would like to restrict the ascription of cardinal numbers to concepts for which it is impossible to go on and individuate further instances.[22] To make this response formally precise, we must add to our basic language $\mathcal{L}$ a new modal operator $\natural$ which ensures that any subformula to which it applies is evaluated at the previous world to have been considered.[23] The operator $\natural$ thus undoes the effect of the innermost modal operator whose scope governs this occurrence of $\natural$. (For instance, $\Box\Diamond\exists x \natural\neg\exists y(y = x)$ says that relative to any world there is another world at which there is an object that does not exist at the previous world.) With this new operator at our disposal we say that a concept F is *rigid* at a world w iff F has precisely the same instances at any world later than w; that is, iff the following holds at w:

$$\Box\forall x(Fx \leftrightarrow \natural(Fx \wedge \exists y(y = x)))$$

Let '$\textsc{Rgd}(F)$' abbreviate the claim that F is rigid. With this terminology in place my claim is that at a coherent world w we can assign cardinal numbers to precisely those concepts that

[22] This response makes formally precise an earlier suggestion due to [Wright, 1999].

[23] This corresponds to the operator $\downarrow$ of [Hodes, 1984], pp. 425-6. A closely related operator is discussed in [Parsons, 1983], pp. 333-5.

are rigid at w:

(N∃) $$\textsc{Rgd}(F) \leftrightarrow \Diamond\exists x\, \textsc{Num}(F, x)$$

We also want to express the claim that cardinal numbers are individuated with respect to one-to-one correspondence. Since the only concepts to have cardinal numbers are the rigid ones, it suffices to consider the *present* instances of a concept. So let '$F \approx G$' be the usual non-modal second-order formalization of the claim that the Fs and the Gs can be one-to-one correlated. The desired claim can then be formalized as

(N=) $$\textsc{Num}(F, x) \wedge \textsc{Num}(G, y) \rightarrow (x = y \leftrightarrow F \approx G).$$

Note that the relation $\approx$ is stable on all and only rigid concepts.[24]

Appendix

I will now provide an example of how Theorem 3 can be applied. Since this example is developed in [Linnebo, 2006], some details will be omitted.[25] We begin with a second-order set theory that allows for urelements. The primitive predicates are '$x = y$', '$x \in y$', and '$\textsc{Ppty}(F, x)$'. I will also use '$x\ \eta\ y$' as an abbreviation for the claim that x has the property y by adopting

(Def-η) $$x\ \eta\ y \leftrightarrow \Diamond\exists F(\textsc{Ppty}(F, y) \wedge Fx).$$

Using the expressive power of set theory we can formulate a condition '$\textsc{Old}_a(x)$' which says that x is definable from some parameter a by means of certain basic operations. We can also prove that this condition is true only of entities individuated no later than a.

Consider now the formula '$x\ \eta\ y$', where 'x' is easily seen to occur in a non-demanding position and 'y' in a demanding position. This means that the condition '$x\ \eta\ y \wedge \textsc{Old}_a(y)$' is predicative and thus suitable for comprehension. Let $\psi_a(x, y)$ abbreviate this condition. By careful choice of the parameter a we can use this condition to individuate increasingly complex properties. First we let a be the property of set membership, that is, the property

[24]This ensures that there is no conflict between (N∃)—which says that only rigid concepts can be assigned numbers—and (A∃)—which says that all concepts can be assigned abstracts: for this latter also requires the relevant equivalence relation to be stable.

[25]A related construction is given in [Fine, 2005a].

definable by the predicative condition '$x \in y$'. Then $\psi_a(x, y)$ individuates a property e_0. Next we let a be e_0 and individuate a more complex property e_1. This process can be continued into the transfinite by letting e_λ be the set $\{e_\gamma : \gamma < \lambda\}$ for any limit ordinal λ. In the mentioned article I show that the resulting hierarchy of theories has the following interesting property: The theory at every stage is precisely the theory required to develop the semantics of the theory at the previous stage in a way that allows all of its quantifiers to have absolutely universal range.

The machinery of Section 7 provides the resources needed to show that all these theories have models. (A slightly different route is provided in [Linnebo, 2006].) Let D be an abstraction domain whose size is some inaccessible cardinal κ. Let *ZFCU* be ordinary ZFC set theory modified so as to allow for urelements. (All we do is relax the axiom of extensionality so that it only requires that *sets* be identical when they have the same elements.) We can then construct a frugal individuation report I_0 on D which gives us a model of ZFCU. (Basically, I_0 represents every plurality of less than κ objects from D as forming a set.) Theorem 1 then enables us to extend this model to a Kripke model that also validates the theories of properties described above. As in the proof of Corollary 1, all we need to do is verify that the comprehension axioms come out true. I claim that this is so under the individuation report I_κ. To see this, let ϕ be a predicative condition. Some of its parameters may be sets. But since all sets are of cardinality $< \kappa$, and since κ is inaccessible (and *a fortiori* regular) there is a $\lambda < \kappa$ such that all of ϕ's parameters are individuated under I_λ. But then ϕ individuates a concept under $I_{\lambda+1}$.

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