Burgess on Plural Logic and Set Theory

Øystein Linnebo
University of Bristol

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Abstract

John Burgess (Burgess, 2004) combines plural logic and a new version of the idea of limitation of size to give an elegant motivation of the axioms of ZFC set theory. His proposal is meant to improve on earlier work by Paul Bernays in two ways. I argue that both attempted improvements fail.

John Burgess (Burgess, 2004) combines plural logic and a new version of the idea of limitation of size to give an elegant motivation of the axioms of ZFC set theory. He proposes that some things form a set just in case they are not "too many" in a precise sense that he articulates. This proposal is supposed to improve on earlier work by Paul Bernays (Bernays, 1976) in two ways. Firstly, where Bernays uses second-order quantifiers ranging over classes, Burgess uses plural quantifiers. In this way he attempts to avoid a problematic commitment to classes and to better capture the attractive Cantorian idea that a set is the result of collecting many objects into one. Secondly, Burgess aims to provide a better motivation for the most distinctive axiom of this approach to set theory, namely the *reflection principle*. He aims to do this by showing how a version of this principle can be motivated from the idea of limitation of size.

In this note I raise some concerns about Burgess's proposed motivation of the axioms of set theory. I develop two arguments that Burgess's proposal also commits him to the view that every plurality forms a set, which in his system leads straight to contradiction. To avoid my two arguments, I argue that Burgess must surrender his proposed improvements on the work of Bernays.

1 Burgess's proposal

In this section I first describe Burgess's proposal and then explain how this proposal is supposed to improve on the work of Bernays.

Burgess's proposal is based on a *plural logic*. In addition to the familiar singular vocabulary of ordinary first-order logic, the language of plural logic contains plural variables such as xx and yy, plural quantifiers such as $\exists xx$ and $\forall yy$ (read as "there are some things xx" and "whenever there are some things yy"),[1] and a two-place logical predicate $\prec$ (whose first and second argument-places are respectively singular and plural, and where $u \prec xx$ is read as "u is one of xx"). Burgess also defines the symbol '==' as follows:

(Def-==) $$xx == yy \leftrightarrow \forall u(u \prec xx \leftrightarrow u \prec yy)$$

The theory of plural logic adds to a complete axiomatization of singular first-order logic an axiom scheme of plural comprehension

(P-Comp) $$\exists xx \forall u[u \prec xx \leftrightarrow \Phi(u)]$$ [2]

and an axiom scheme of extensionality for pluralities[3]

(P-Ext) $$xx == yy \rightarrow (\Phi(xx) \leftrightarrow \Phi(yy)).$$

The instances of both schemes may contain free (singular and plural) variables, known as (respectively singular and plural) *parameters*. Such instances are tacitly understood as being prefaced by universal quantifiers binding these free variables.[4] This convention will govern all axioms to be discussed in this note.

Burgess then defines a *plural set theory* based on this plural logic. This theory is named *BB* in honor of George Boolos (who inspired its logical aspects) and Paul Bernays (who inspired its non-logical aspects).[5] The language of BB adds to the language of plural logic the following two non-logical primitives: $ßu$ for 'u is a set', and $u \equiv xx$ for 'u is a set whose elements are all and only the things xx'. Given these primitives, Burgess defines the relation

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[1] Burgess writes these quantifiers as $\exists\exists xx$ and $\forall\forall xx$. I prefer to follow the tradition of second-order logic and type theory and let the type of the variable to which $\exists$ or $\forall$ is attached indicate the type over which the resulting quantifier ranges.

[2] As usual, Φ must not contain xx free. Moreover, this scheme and the next are understood in an open-ended manner so as to allow Φ to be a formula of any extended language. Henceforth I won't mention these facts. Note also that this comprehension scheme allows for empty pluralities; for discussion see Burgess, 2004, p. 197. Unless otherwise stated, all page references will henceforth be to this article.

[3] I sometimes use the singular locution 'plurality' when talking about some things. I do this solely for reasons of style and simplicity; the reader may translate back into genuinely plural locutions if she desires.

[4] See p. 196.

[5] See Boolos, 1984 and Boolos, 1985, and Bernays, 1976. For another version of plural set theory based on the idea of limitation of size, see Pollard, 1996.

of element to set as follows:

(Def-$\in$)
$$v \in u \leftrightarrow \text{ß}u \land \exists xx(u \equiv xx \land v \prec xx)$$

The theory of BB adds to the theory of plural logic the following non-logical axioms.

First there is the axiom of Heredity:

(Her)
$$\text{ß}u \leftrightarrow \exists xx(u \equiv xx)$$

This axiom ensures that whenever a domain contains a set, it also contains all elements of this set. We will see shortly that this apparently innocent claim is crucial to the second of Burgess's attempted improvements on Bernays.

Next, the criterion of identity for sets is given by the axiom of Extensionality for sets:

(Ext)
$$u \equiv xx \land v \equiv yy \rightarrow (u = v \leftrightarrow xx == yy)$$

This ensures that when some things xx form a set, the identity of this set is determined by which things xx encompass. Burgess observes that Extensionality is "less a substantive assumption than a partial explication of the *concept* of set" (p. 199).

Then there is the axiom of Separation, which says that whenever some things form a set, so do any other things that are among the first things:

(Sep)
$$\forall w(w \prec yy \rightarrow w \prec xx) \rightarrow [\exists u(u \equiv xx) \rightarrow \exists v(v \equiv yy)]$$

Separation is an obvious ingredient of any version of the idea of limitation of size. For it is indisputable that if some things are among some other things, then the former things cannot be more numerous than the latter things.

Finally, there is the most distinctive and interesting axiom (scheme) of BB, namely the reflection principle. Burgess motivates this axiom scheme by starting with a rough version of the idea of limitation of size, which he then gradually refines. We begin with the idea that some things xx form a set unless they are indescribably many. This can be refined as the principle that some things xx form a set unless any statement Φ that holds of them fails to describe how many they are. This can in turn be refined as the principle that some things xx form a set unless "any statement Φ that holds of them continues to hold if reinterpreted to be not about all of them but just about some of them, few enough to form a set" (p. 205). Consider now the instance where xx are all objects. On pain of paradox, this plurality cannot form a set. We thus get the principle that any statement Φ that holds continues to hold if reinterpreted to be just about the elements of some set t. Note that this argument makes absolutely no mention of what vocabulary may figure in Φ. This will become important in the next two sections.

What remains in order to arrive at the reflection principle is to explain what it is to reinterpret a statement Φ to be just about the elements of some set t. According to Burgess, to reinterpret Φ in this way is just to restrict all its quantifiers to t. Let '$uu \in\in x$' abbreviate '$\forall v(v \prec uu \rightarrow v \in t)$'. To restrict the quantifiers of Φ to t, what we do is replace every occurrence of a singular universal quantifier $\forall u$ with '$\forall u(u \in t \rightarrow \ldots)$' (which in turn we abbreviate as '$(\forall u \in t)(\ldots)$'), replace every occurrence of a plural universal quantifier $\forall uu$ with '$\forall uu(uu \in\in t \rightarrow \ldots)$' (which we abbreviate as '$(\forall uu \in\in t)(\ldots)$'), and likewise for the existential quantifiers. It is very important that all formulas be written out in primitive notation before this process of restricting quantifiers is applied.[6] Let Φ^t be the result of applying this process to Φ. Then the reflection principle can be formulated as follows:

(Refl) $$\Phi \rightarrow \exists t \Phi^t$$

Recall that each instance of this axiom scheme is tacitly understood as being prefaced by universal quantifiers binding each of its free variables.

Let ZFU be ordinary ZF set theory modified so as to allow for urelements. This modification consists of restricting the set-theoretic axiom of extensionality to sets; that is, to let the axiom say that, on the assumption that x and y are sets, they are identical just in case they have precisely the same elements. Building on Bernays, 1976, Burgess then proves the following theorem.

Theorem 1 BB entails all the axioms of ZFU except Foundation.

This is an amazing result because it shows that (against the background of plural logic) a theory motivated entirely by a natural interpretation of the idea of limitation of size gives rise to almost all of ordinary set theory. In particular, the result traces all the disparate set existence axioms of ZF back to a unique source and in this way provides a unified explanation of what sets there are. Less importantly for present purposes, BB also entails the existence of "small large cardinals" such as inaccessible cardinals and Mahlo cardinals. Moreover, Burgess shows how Foundation can be obtained by restricting all quantifiers to well-founded sets, and how Choice can be obtained by adopting a choice principle in the underlying plural logic. By thus accounting for the two axioms "missing" from the above theorem, Burgess has accounted for all of ZFCU (that is, standard ZFC set theory modified as above to allow for urelements).

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[6] See p. 207.

A crucial step in the proof of Theorem 1 is to establish three strengthenings of (Refl):

(Refl$^+$) $$\Phi \to \exists t(\text{ß}t \wedge \Phi^t)$$

(Refl^{++}) $$\Phi \to \exists t(\text{ß}t \wedge t \text{ is transitive} \wedge \Phi^t)$$

(Refl^{+++}) $$\Phi \to \exists t(\text{ß}t \wedge t \text{ is supertransitive} \wedge \Phi^t)$$

(A set is *transitive* if it contains every element of each of its elements. It is *supertransitive* if, in addition to being transitive, it contains every subset of each of its elements.) As will become important below, the steps from the first strengthening to the second and from the second to the third make essential use of respectively the axioms of Heredity and Separation.[7]

We are now in a position to explain Burgess's two intended improvements on the work of Bernays. Firstly, where Bernays uses second-order quantifiers ranging over classes, Burgess uses plural quantifiers. In this way he attempts to avoid an undesirable commitment to classes, which he takes to be "set-like entities that in some mysterious way fail to be sets."[8] The use of plural quantifiers also enables Burgess to better capture the attractive Cantorian idea that a set is the result of collecting many objects into one.[9] Secondly, Burgess aims to provide an intuitive and systematic motivation of the axioms of set theory by means of his novel interpretation of the idea of limitation of size.[10] By contrast, Bernays, whose goal is purely mathematical, makes no attempt to provide any such a motivation. But Burgess's second goal faces a slight complication. For where Bernays uses the somewhat complicated reflection principle (Refl^{++}) as his starting point, Burgess's account only motivates the simple reflection principle (Refl) and its trivial strengthening (Refl$^+$). But this complication is cleverly dealt with by means of Burgess's strategy for obtaining (Refl^{++}) from (Refl$^+$). As mentioned above, the axiom (Her) plays an essential role in this strategy.

2 The Problem of Plural Parameters

Although Burgess's account motivates the axioms of set theory in a natural and elegant way, I will now give two arguments that this account also motivates the view that every plurality forms a set. In a theory such as Burgess's with an unrestricted plural comprehension scheme (P-Comp), this view leads straight to contradiction. To see this, let rr be the sets that are

[7] See p. 208. The trick is to apply (Refl) to the conjunction of Φ with various provable formulas, including (Her) and (Sep).

[8] See p. 200. See also Burgess, 2005, pp. 210-214, where it is argued that the only legitimate interpretation of second-order logic is Boolos's in terms of plural quantification.

[9] See p. 193.

[10] See especially pp. 204-205.

not elements of themselves, and let r be the set that they form. Then r is an element of itself just in case it isn't.

My first argument concerns the behavior of plural parameters under the reinterpretations that Burgess uses to motivate his reflection principle. A "reinterpretation," in the relevant sense, is just a matter of restricting quantifiers: the reference of parameters and the interpretation of predicates (that is, their satisfaction conditions) are kept fixed. To make this clear, I will refer to such reinterpretations as *quantifier-reinterpretations* or *q-reinterpretations* for short. The domain associated with any q-reinterpretation of a formula Φ must thus contain the referents of all parameters occurring in Φ. For instance, the domain associated with any q-reinterpretation of 'Socrates is mortal' must contain Socrates. Likewise, the domain associated with any q-reinterpretation of 'Socrates and Plato agreed' must contain both Socrates and Plato. This means that the motivation Burgess provides for his reflection principle (Refl) also motivates a stronger reflection principle. For simplicity, I will only state this stronger principle for a formula Φ whose only parameter is uu; the general case is an obvious extension.

$$\text{(Refl}'\text{)} \qquad \forall uu[\Phi \rightarrow \exists t(uu \in\in t \wedge \Phi^t]$$

But given this reflection principle, we can easily prove that every plurality forms a set. Let aa be an arbitrary plurality, and let Φ be a formula containing a plural parameter referring to aa. Φ can then be q-reinterpreted as being about a domain that forms a set t. But this set must contain the referents of all parameters occurring in this formula, including the plural ones—as made explicit in (Refl′). It thus follows that aa must be contained in the set t. By Separation, we then get that aa form a set. Finally, since aa were arbitrary, it follows that every plurality forms a set. I will refer to this as *the Problem of Plural Parameters*. I will now consider three objections to this argument.

Firstly, it may be objected that even if my argument works, this just shows that the reflection scheme must be restricted to formulas that don't contain plural parameters. The most natural such restriction would be to limit the reflection principle to *sentences*, that is, to formulas containing neither singular nor plural parameters. However, the resulting reflection principle would be too weak for Burgess's purposes, as his account uses reflection on formulas containing singular parameters.[11] Consider the proof of the axiom of Pairing that Burgess adopts from Bernays. Given any two objects a and b, we have the following logical truth

[11] This use of singular parameters can be eliminated in favor of plural ones. But the use of parameters cannot be eliminated altogether.

(with singular parameters a and b):

(1) $$\exists u(u = a) \wedge \exists u(u = b)$$

By reflection on this formula we get $\exists t(a \in t \wedge b \in t)$. By Separation, this yields a set whose sole elements are a and b.

Can the first objection instead be implemented by drawing a line between singular and plural parameters, allowing reflection on formulas containing the former but not the latter? One problem with this implementation is that Burgess himself needs to allow reflection on formulas with plural parameters in order to prove the axiom of Replacement. For the proof of this axiom, Burgess defers to Bernays, whose proof makes essential use of reflection on a formula with a free second-order parameter.[12] Translated into Burgess's plural logic, this corresponds to reflection on a formula with a free plural parameter. But even if this problem could somehow by bypassed,[13] the present proposal would still face another problem, namely that it would be completely *ad hoc*. For the motivation that Burgess offers for the reflection principle provides no basis for such a distinction between singular and plural parameters: nothing in the proposed motivation is in any way sensitive to this distinction.

Secondly, it may be objected that the notion of reinterpretation relevant to Burgess's motivation of the reflection principle isn't that of a q-reinterpretation but rather some other notion. If so, no indication of what this alternative notion is is to be found the article. But more seriously, it is hard to see how some notion other than that of a q-reinterpretation could be better suited to the task of motivating the reflection principle. The motivating idea is that any property had by the entire universe is already had by some set-sized sub-universe. But this idea is entirely a matter of the ranges of the quantifiers. The predicates and parameters that are used to define the relevant properties must therefore be left unchanged.

Thirdly, it may be objected that I have misunderstood how plural parameters behave under q-reinterpretations. Is it really necessary that all the referents of a plural parameter be contained in the domain of any q-reinterpretation? I think it is. I think this follows from the nature of plural reference and from the identity condition for pluralities. Consider the natural language expressions in terms of which the plural variables were explained, namely plural pronouns and plural demonstratives. The semantic task of a plural term like 'these' is

[12] See respectively Burgess, 2004, p. 209 and Bernays, 1976, pp. 133-135.

[13] An anonymous referee suggested the following way of bypassing the problem. In addition to reflection, Burgess's theory BB contains Separation as a further axiom independently motivated by limitation of size. But instead of adopting Separation as a further axiom, we could adopt a plural version of Replacement, which is also naturally motivated by limitation of size. We would then no longer need to obtain Replacement from the reflection scheme, which would eliminate the only need for reflection on formulas with plural parameters. And as is well know, the axiom of Separation is an immediate consequence of Replacement.

to refer directly and rigidly to some things, just as the semantic task of a singular term like 'this' is to refer directly and rigidly to an individual thing. Moreover, the particular things to which the word 'these' refers (in some context of use) would not be the things they are unless they encompass precisely the things that they in fact encompass. Were we to remove some of these things, we would no longer be talking about the same things but about *some other things*, which include some but not all of the original things. Thus, so long as the interpretation of a plural referring term remains fixed, this term refers to precisely the same objects if it refers at all. For instance, if our domain contains Plato but not Socrates, then the plural term 'Socrates and Plato' must be deemed non-referring, just as the singular term 'Socrates'.

By contrast, a concept or the extension of a predicate can plausibly be taken to vary with the domain that we are considering. For instance, Socrates need not fall under the concept of being a philosopher or be a member of the extension of '...is a philosopher' when we are concerned with a domain that does not contain Socrates. Inspired by this observation, we may attempt to reformulate Burgess's proposal in terms of second-order logic ranging over concepts or related entities rather than in terms of plural logic. A plausible argument can then be given that this reformulation of Burgess's proposal is immune to my present argument. However, this reformulation would amount to giving up on the first of Burgess's two proposed improvements on Bernays.

The conclusion of this section is thus that the only promising strategy for avoiding the Problem of Plural Parameters requires one to give up the first of Burgess's two improvements over Bernays. More precisely, the plural quantifiers must be replaced by second-order quantifiers interpreted in a sufficiently "conceptual" way to ensure the success of the last of the three objections to the Problem of Plural Parameters. I make some comments about the viability of this strategy in Section 4.

3 The Problem of Additional Primitives

I now turn to my second argument that Burgess's motivation of the axioms of set theory also motivates the undesired claim that every plurality forms a set. This argument, which is independent of the first, is based on the introduction of additional primitive expressions into our language. The introduction of new primitives is something that Burgess's own account crucially relies on.[14] For recall that Burgess decides to treat the notion ß of being a set as

[14]Burgess appears to be aware of the sort of challenge presented in this section. For he writes on p. 207 that the "particular choice of primitives used here is [...] another point that might be open to skeptical challenge." Unfortunately, Burgess does not say anything about how such a challenge should be answered.

a new primitive subject to the axiom of Heredity, although this axiom would more naturally have been regarded as a definition of a non-primitive notion ß. This decision is essential to the second of Burgess's two improvements over Bernays mentioned at the end of Section 1. For the axiom of Heredity plays a crucial role in his derivation of (Refl^{++}) from (Refl^{+}). And it is only this latter form of reflection, not the former, that receives any direct support from Burgess's development of the idea of limitation of size.

I begin by showing how the introduction of a new primitive predicate that stands to pluralities the way the identity predicate stands to ordinary objects leads to a collapse of pluralities to sets. Let $\approx$ be a new primitive predicate which takes two plural arguments and which is characterized by the following axiom:

(2) $$xx \approx yy \leftrightarrow \forall u(u \prec xx \leftrightarrow u \prec yy)$$

Since $\approx$ is an equivalence relation on pluralities that supports Leibniz's Law, it does indeed stand to pluralities the way identity stands to ordinary objects. Equipped with this new predicate and the associated axiom (2), we can prove that every plurality forms a set. For whenever aa are some things, it is a theorem that $\exists xx(xx \approx aa)$. Reflection on this theorem yields $(\exists t)(\exists xx \in\in t)(xx \approx aa)$. Since xx can bear $\approx$ to aa only if xx encompass precisely the same objects as aa (not just in some restricted interpretation but absolutely), this means that the things aa are contained in a set and hence by Separation form a set. Since aa were arbitrary, this means that every plurality forms a set.

Note that it is essential to this argument that $\approx$ be regarded as a new primitive. To see this, consider what happens when the reflection principle is applied to the formula $\exists xx(xx == aa)$. When the defined predicate $==$ is eliminated in favor of primitive notation, reflection on the resulting formula yields the entirely innocent claim that there is a set t containing all objects among aa that are elements of t:

(3) $$(\exists t)(\exists xx \in\in t)(\forall u \in t)(u \prec xx \leftrightarrow u \prec aa)$$

Note also that the above argument depends on allowing reflection on formulas containing plural parameters. But this dependence can be eliminated if desired. To see how, recall the plural comprehension scheme:

(P-Comp) $$\exists xx \forall u(u \prec xx \leftrightarrow \Psi(u))$$

Each instance of this scheme contains a subformula $\forall u(u \prec xx \leftrightarrow \Psi(u))$ that uniquely characterizes a plurality xx. Corresponding to each such subformula we can introduce a new predicate P_Ψ, governed by an axiom of the form

$$\forall xx[P_\Psi(xx) \leftrightarrow \forall u(u \prec xx \leftrightarrow \Psi(u))]. \tag{4}$$

When we add (4) as an axiom to a theory with a plural comprehension axiom for Ψ, we can prove that $\exists xx\, P_\Psi(xx)$. Reflection on this theorem then yields:

$$(\exists t)(\exists xx \in\in t)P_\Psi(xx) \tag{5}$$

From (4) and (5) it then follows that that the plurality defined by Ψ is contained in a set:

$$(\exists t)(\exists xx \in\in t)(\forall u)(u \prec xx \leftrightarrow \Psi(u)) \tag{6}$$

By Separation we finally conclude that this plurality forms a set. In fact, since Burgess allows plural comprehension on a formula Ψ that says that xx include every object there is, it follows by Separation that every plurality forms a set.

It may be objected to the two collapse arguments presented in this section that we are not allowed to adopt new primitive predicates such as $\approx$ and P_Ψ and axioms that characterize these primitives. The most obvious way of spelling out this objection would be as an insistence that any notion that *can* be introduced as a defined notion *should* be so introduced (if at all) and not as a new primitive subject to some characterizing axiom. But this hard line isn't available to Burgess, whose own account crucially relies on treating the notion ß as a new primitive characterized by the axiom of Heredity, although this notion would more naturally have been introduced by a definition. So if the present objection is to succeed, a principled line will have to be drawn between notions such as ß, which the objector is willing to accept as primitives, and notions such as $\approx$ and P_Ψ, which the objector is unwilling thus to accept. Unfortunately, the motivation that Burgess provides for the reflection principle provides no indication of how such a line might be drawn.

To investigate if someone objecting to the two collapse arguments can do better, we need a deeper understanding of what is going on when a notion is treated as a primitive. Let Ψ be a formula among whose free singular and plural variables (or "parameters" as we are also calling them) are respectively $x_1, \ldots, x_m$ and $uu_1, \ldots, uu_n$. Let $\mathbf{x}$ and $\mathbf{uu}$ abbreviate these two sequences of variables. Suppose we introduced a primitive predicate R for this relation, governed by the axiom

$$\forall \mathbf{x} \forall \mathbf{uu}[R\mathbf{x}, \mathbf{uu} \leftrightarrow \Psi(\mathbf{x}, \mathbf{uu})]. \tag{7}$$

Reflection on $\Phi \wedge (7)$ then yields

$$(8) \qquad \exists t[\Phi^t \wedge (\forall \mathbf{x} \in tt)(\forall \mathbf{uu} \in\in t)(R\mathbf{x}, \mathbf{uu} \leftrightarrow \Psi(\mathbf{x}, \mathbf{uu})^t]$$

which when combined with (7) yields

$$(9) \qquad \exists t[\Phi^t \wedge (\forall \mathbf{x} \in t)(\forall \mathbf{uu} \in\in t)(\Psi(\mathbf{x}, \mathbf{uu}) \leftrightarrow \Psi(\mathbf{x}, \mathbf{uu})^t)].$$

So if unconstrained, the trick involved in Burgess's decision to treat ß as a primitive will allow us to require, for any formula Ψ, that the domain t introduced by a reflection axiom can be chosen so as to make Ψ *absolute* for t.[15]

This analysis shows that the question when (in the presence of the reflection principle) it is permissible to regard a notion as a primitive governed by some characterizing axiom can be reduced to the question when it is permissible to require that the domain t be absolute with respect to the notion in question. Does this help the objector distinguish between ß on the one hand and $\approx$ and P_Ψ on the other? At first sight things look good for the objector. For as we have seen, to insist that ß be absolute for t is just to insist that t be transitive. In contrast, to insist that P_Ψ be absolute for t is to insist that the formula Ψ that figures in a plural comprehension axiom also defines a set. So where the first insistence appears to be innocent, the second appears to be anything but.

However, on a closer examination we see that the present development of the objection completely reverses the intended order of explanation. To be entitled to treat ß as a primitive governed by the axiom of Heredity, the objector now needs some *prior guarantee* that the domain t can be chosen so as to be transitive. But this claim was supposed to *follow* from the account based on the reflection principle, not to be something *presupposed* by it! Likewise, what entitles the objector to regard as illegitimate the requirement that P_Ψ be absolute for t? This verdict will follow only if the objector has some prior guarantee that the formula Ψ doesn't define a set. But Burgess's account was supposed to *explain* what sets there are, not *presuppose* such an explanation. The objection therefore fails to make out the required distinction between notions that may and may not be regarded as primitives.

What conclusions can be drawn from the discussion of this section? The most promising strategy for avoiding the Problem of Additional Primitives would probably be to give up Burgess's trick of treating ß as a primitive governed by (Her). This would enable him to disallow the introduction of *any* new primitives governed by new axioms, which would block

[15]In ordinary set theory a formula Ψ whose free variables are $\mathbf{x}$ is said to be *absolute for t* iff $(\forall \mathbf{x} \in t)(\Psi(\mathbf{x}) \leftrightarrow \psi(\mathbf{x})^t)$. The notion of absoluteness I am using here is an obvious adaptation of this standard notion to the context of plural set theory.

all of the collapse arguments presented in this section. Unfortunately, this strategy is highly problematic. To begin with, the strategy would undermine Burgess's proof of (Refl^{++}) from (Refl^{+}), which in turn would threaten his attempted motivation of the axioms of set theory. For recall that his development of the idea of limitation of size only motivates the weaker reflection principle (Refl^{+}), not the stronger one (Refl^{++}), which was Bernays' starting point.

Perhaps this problem can be avoided by tweaking the interpretation of the idea of limitation of size so that it also motivates the stronger reflection principle (Refl^{++}). But even if this could be pulled off, Burgess would not be off the hook. For although this would show that no additional primitives *need* be accepted, it would do nothing to establish that additional primitives *may not* be accepted. For as we have seen, the motivation that Burgess provides for the reflection principle makes no mention of the expressive resources of the language. On the contrary, the motivation appears to be completely insensitive to the nature and extent of the expressive resources. So if the motivation works at all, it will work equally well for languages that contain additional primitives. This problem threatens to undermine the second of Burgess's two improvements on Bernays, namely his use of the idea of limitation of size to provide a natural and systematic motivation of the axioms of set theory. The only satisfactory solution would be to provide a better motivation of the reflection principle that explains what expressive resources may figure in formulas to which the reflection principle is applied. But it is completely unclear what such a motivation would look like.

4 Back to Bernays

Let's take stock. I have given two arguments that Burgess's motivation of the axioms of set-theory also motivates the undesired view that every plurality forms a set. I have also claimed that the first argument can be avoided only by giving up Burgess's first attempted improvement on Bernays, and that the second argument can (most likely) be avoided only by giving up the second attempted improvement. Combining these conclusions, there appear to be only two options. The *heroic option* is to accept that every plurality forms a set, while attempting to restore consistency by means of other adjustments to the theory. The *conservative option* is to give up on Burgess's attempted improvements and go back to Bernays. I consider each option in turn.

What are the prospects for the heroic option? As we have seen, in the presence of the unrestricted plural comprehension scheme (P-Comp), the view that every plurality forms a set leads straight to contradiction. So if this option is to be viable, the plural comprehension scheme must be restricted in some way. Many will no doubt regard any such restriction as sufficiently unpalatable to warrant an immediate rejection of this option. I myself do not find this so obvious.[16] Fortunately, there is no need to examine this matter here, as there is a compelling technical reason why accepting the collapse of pluralities to sets is incompatible with the reflection principle.[17] Assume we accept that every plurality forms a set. This means that we accept $\forall uu \exists x(x \equiv uu)$. Reflection on this formula then gives us $(\exists t)(\forall uu \in\in t)(\exists x \in t)(x \equiv uu)$. But this enables us to prove that there is a set t containing all of its own subsets: $\exists t \forall x(x \subseteq t \rightarrow x \in t)$. But this leads straight to contradiction. To see this, let $a = \{u \in t \,|\, u \notin u\}$. Since a is a subset of t and thus also an element of t, it follows that $a \in a \leftrightarrow a \notin a$.

This leaves only the conservative option of giving up on Burgess's attempted improvements and instead going back to Bernays. To assess the prospects for this option, we need to consider the two problems discussed in the previous two sections. I concluded at the end of Section 2 that the only promising strategy for avoiding the Problem of Plural Parameters involves giving up the first of Burgess's two improvements over Bernays. More specifically, I claimed that we must replace the plural quantifiers with second-order quantifiers ranging over concepts or concept-like entities so as to ensure that when the domain is restricted, the semantic value of second-order parameters must be restricted accordingly. It is unclear to me whether Bernays' own interpretation of the second-order quantifiers as ranging over classes meets this requirement. More generally, it is unclear whether there is *any* interpretation of the second-order quantifiers which *both* satisfies this requirement *and* supports full impredicative second-order comprehension.[18] I cannot attempt to settle this question here. I content myself with the observation that although replacing plural logic with traditional second-order logic goes a long way towards solving the Problem of Plural Parameters, it is not obvious that it provides an entirely satisfactory response.

I concluded at the end of Section 3 that the Problem of Additional Primitives forces Burgess to give up the second of his attempted improvements on Bernays (or at least that this will be so in the absence of a better motivation of the reflection principle that manages to explain which expressive resources are permitted in the formulas to which the principle is applied). Assume we respond to this problem by giving up on Burgess's attempt to provide an intuitive motivation of the reflection principle from the idea of limitation of size. We will then be free to *stipulate* the language to which the principle may be applied. However, lowering our ambition in this way will bring with it the danger of making the reflection principle seem

[16] See Linnebo, .

[17] Thanks to Ignacio Jané for pointing this out to me.

[18] For a skeptical view, see Parsons, 1974. For a more optimistic view, see Gödel, 1944. Also relevant in this connection is Bernays, 1935, where it is argued that the justification for impredicative second-order comprehension draws on broadly set-theoretic ideas. But Bernays, 1976, pp. 137-8 shows that reflection for formulas with bound second-order variables implies impredicative comprehension.

ad hoc. Whether this is a problem for Bernays will depend on the precise nature of his goal, which cannot be examined here.

I conclude that more work needs to be done before a return to Bernays will provide an entirely satisfactory solution to the problems discussed in this note.[19]

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