The Individuation of the Natural Numbers

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1 Introduction

It is sometimes suggested that criteria of identity should play a central role in an account of our most fundamental ways of referring to objects. The view is nicely illustrated by an example due to (Quine, 1950). Suppose you are standing at the bank of a river, watching the water that floats by. What is required for you to refer to the river, as opposed to a particular segment of it, or the totality of its water, or the current temporal part of this water? According to Quine, you must at least implicitly be operating with some criterion of identity that informs you when two sightings of water count as sightings of the same referent. For unless you have at least an implicit grasp of what is required for your intended referent to be identical with another object with which you are presented, you have not succeeded in singling out a unique object for reference.

This view goes back at least to Frege. In his *Foundations of Arithmetic* Frege first argues that the natural numbers are abstract objects. Then he asks how these objects "are given to us" ((Frege, 1953), §62). Unlike ordinary concrete objects, we cannot have any "ideas or intuitions" of the natural numbers. How then do we manage to refer to natural numbers? Frege answers as follows.

> If we are to use the symbol a to signify an object, we must have a criterion for deciding in all cases whether b is the same as a, even if it is not always in our power to apply this criterion. (*ibid.*)

This passage and surrounding ones show that Frege took criteria of identity to play a very important role in an account of reference to the natural numbers.

The view that criteria of identity play a central role in our most fundamental forms of reference is an attractive one. As Quine's example nicely brings out, the idea that reference to an object crucially involves an ability to distinguish the actual referent from other candidate referents enjoys great pre-theoretic plausibility. Another attraction of the view is its great generality. Since the notion of a criterion of identity is applicable to all kinds of objects, this approach to the problem of reference is applicable not just to concrete objects but also to abstract ones. This is a major advantage over competing approaches. For the language of mathematics abounds with apparent cases of reference to mathematical objects. And this language seems to succeed in expressing all kinds of truths. This phenomenon calls for an explanation, not a dismissal. But many approaches to the problem of reference are unable to accommodate reference to abstract objects. For instance, if the relation of reference is based entirely on causal relations, there can be no reference to abstract objects.[1] By contrast, since criteria of identity are found in the abstract realm as well as in the concrete, the approach in question extends naturally to abstract objects.

The aim of this paper is to investigate how an approach to the problem of reference which gives pride of place to criteria of identity can be applied to the natural numbers. Other than the informal considerations just adduced, I will not attempt any direct defense of this kind of approach. My hope is rather that my investigation will produce an account of reference to the natural numbers which is attractive enough to provide indirect support for this approach.

The paper is organized as follows. I begin by clarifying the notions of individuation and criterion of identity, which play a central role in my investigation. Then I explain two competing criteria of identity which have been argued to play a central role in reference to the natural numbers. One criterion regards the natural numbers as *cardinal numbers*, individuated by the cardinalities of the collections that they number. This account is favored by classic logicists such as Frege and Russell and by their followers.[2] The other criterion regards the natural numbers as *ordinal numbers*, individuated by their positions in the natural number sequence. This account is favored by many constructivists and non-eliminative structuralists.[3] Next I outline some arguments in favor of the ordinal account. I end by developing the ordinal

[1]Indeed, according to the standard definition, an object is *abstract* just in case it is non-spatiotemporal and does not stand in causal relations.

[2]See for instance (Frege, 1953), (Russell, 1919), (Wright, 1983), and (Hale and Wright, 2001).

[3]The account is defended by ((Dedekind, 1996)) and, more recently, by various non-eliminative mathematical structuralists, such as (Parsons, 1990), (Resnik, 1997), and (Shapiro, 1997).

account in more detail and discussing its implications concerning the metaphysical status of the natural numbers.

A broader lesson arising from this article is that the philosophy of mathematics ought to pay more attention to psychology and the philosophy of language than is currently done.[4]

2 Individuation and criteria of identity

My clarification of the notions of individuation and criterion of identity is organized around three important distinctions.

The first distinction is between two senses of the word ‘individuation’—one semantic, the other metaphysical. In the *semantic* sense of the word, to individuate an object is to single it out for reference in language or in thought. The problem discussed above of how this singling out is effected is thus a problem concerning semantic individuation. By contrast, in the *metaphysical* sense of the word, the individuation of objects has to do with “what grounds their identity and distinctness.”[5] Sets are often used as an example. The identity or distinctness of sets is grounded in accordance with the principle of extensionality, which says that two sets are identical iff they have precisely the same elements:

(Ext) $$\textsc{Set}(x) \wedge \textsc{Set}(y) \rightarrow [x = y \leftrightarrow \forall u(u \in x \leftrightarrow u \in y)]$$

The metaphysical and semantic senses of individuation are quite different notions, neither of which appears to be reducible to or fully explicable in terms of the other. My focus here will be on the semantic notion of individuation. I find this notion clearer and more amenable to systematic investigation. In fact, I am not convinced that sufficient sense can be made of the alleged metaphysical notion of individuation.

Next, what is the relation between the semantic notion of individuation and the notion of a *criterion of identity*? According to the approach to the problem of reference with which I will be concerned, there is a very close relation. It is by means of criteria of identity that semantic individuation is effected. To single out an object for reference involves being able to distinguish this object from other possible referents.[6]

[4]Thomas Hofweber’s contribution to this volume draws a similar lesson (although our attempts to heed this lesson yield quite different results).

[5]See e.g. (Lowe, 2003).

[6]See (Lowe, 2003) for a comparison of the metaphysical notion of individuation with a metaphysical notion

The final distinction is between two types of criteria of identity. A *one-level criterion of identity* says that two objects of some sort F are identical iff they stand in some relation R_F:

$$Fx \wedge Fy \to [x = y \leftrightarrow R_F(x, y)] \tag{1L}$$

Criteria of this form operate at just one level in the sense that the condition for two objects to be identical is given by a relation on these objects themselves. An example is the set-theoretic principle of extensionality.

A *two-level criterion of identity* relates the identity of objects of one sort to some condition on entities of another sort. The former sort of objects are typically given as functions of items of the latter sort, in which case the criterion takes the form:

$$f(\alpha) = f(\beta) \leftrightarrow \alpha \approx \beta \tag{2L}$$

where the variables α and β range over the latter sort of item and $\approx$ is an equivalence relation on such items.[7] An example is Frege's famous criterion of identity for directions:

$$d(l_1) = d(l_2) \leftrightarrow l_1 \parallel l_2 \tag{2L-Dir}$$

where the variables l_1 and l_2 range over lines or other directed items. An analogous two-level criterion relates the identity of geometrical shapes to the congruence of things or figures having the shapes in question. Some terminology will be useful. Let's refer to the items over which the variables α and β range as *presentations*. The idea is that these are items that present the objects with whose identity we are concerned. Let's refer to the equivalence relation $\approx$ as a *unity relation*.

My decision to focus on the semantic notion of individuation makes it natural to focus on two-level criteria. For two-level criteria of identity are much more useful than one-level criteria when we are studying how objects are singled out for reference. A one-level criterion provides little assistance in the task of singling out objects for reference. In order to apply a one-level criterion, one must already be capable of referring to objects of the sort in question. By contrast, a two-level criterion promises a way of singling out an object of one sort in terms

of a criterion of identity.

[7] An approach based on two-level criteria of identity is found in (Williamson, 1990), chapter 9.

of an item of another and less problematic sort. For instance, when Frege investigated how directions and other abstract objects "are given to us" although "we cannot have any ideas or intuitions of them" ((Frege, 1953), §62), he proposed that we relate the identity of two directions to the parallelism of the two lines in terms of which these directions are presented. This would be explanatory progress, since reference to lines is less puzzling than reference to directions.

3 The individuation of the natural numbers

How are the natural numbers individuated? The views found in the literature naturally fall into two types: those that take the natural numbers to be individuated as *cardinal numbers*, and those that take them to be individuated as *ordinal numbers*.

According to the former type of view, the natural numbers are individuated by the cardinalities of the concepts or the collections that they number. For instance, our most fundamental way of thinking of the number 5 is as the cardinal measure of quintuply instantiated concepts or five-membered collections. This view naturally corresponds to a two-level criterion of identity. According to Frege, a number is presented by means of a concept which has the number in question, and two concepts F and G determine the same number just in case the Fs and the Gs can be put in one-to-one correspondence. Let $F \approx G$ abbreviate the standard second-order formalization of this requirement. Frege's claim is then that the natural numbers are subject to the following criterion of identity, which has become known as *Hume's Principle*:

(HP) $$\#F = \#G \leftrightarrow F \approx G$$

Similar views have been defended by Russell and the neo-Fregeans Bob Hale and Crispin Wright.[8]

The view that the natural numbers are finite cardinals individuated by (HP) has some attractive features. Many philosophers find it attractive that the view builds the application of the natural numbers as measures of cardinality directly into their identity conditions;

[8]See also Roy Cook's contribution to this volume. I here gloss over some important differences which are irrelevant to our present concerns. In particular, Frege and Russell wanted to reduce (HP) to what they took to be more basic class-theoretic notions. For details see (Frege, 1953), (Russell, 1919), (Wright, 1983), and (Hale and Wright, 2001).

for instance, the number three is individuated as the number that counts all triples. But the most impressive feature is no doubt a technical result known as *Frege's Theorem*. Consider the theory that consists of pure second-order logic and (HP) as a sole non-logical axiom. Frege's Theorem says that this theory and some natural definitions suffice to derive all the familiar axioms of second-order Dedekind-Peano Arithmetic and thus all of ordinary arithmetic.[9]

According to the competing view, the natural numbers are individuated by their ordinal properties, that is, by their position in the natural number sequence. For instance, our most fundamental way of thinking of the number 5 is as the fifth element of this sequence. This view too corresponds naturally to a two-level criterion of identity. A natural number is presented by means of a "counter" or a numeral, which occupies a unique position in a sequence. Two such "counters" or numerals determine the same number just in case they occupy the same position in their respective orderings. For instance, the decimal numeral '5' is equivalent to the Roman numeral 'V' because both occupy the fifth position in their respective orderings. If we symbolize this latter relation by $\sim$, this can be expressed as the following criterion:

(2L-ℕ) $$N(\bar{m}) = N(\bar{n}) \leftrightarrow \bar{m} \sim \bar{n}$$

This criterion of identity will be developed in more detail in Section 5.

Which of these two views on the individuation of the natural numbers is more plausible? In the next two sections I defend the ordinal conception. However, before embarking on this defense I would like to pause briefly to reflect on the question we are discussing. Firstly, does the question presuppose that there really are natural numbers? Since the notion of individuation with which we are concerned is the semantic one, the answer is *no*. Our question is concerned with the features of our conception of natural numbers which are responsible for singling them out for reference. And these features can be investigated regardless of whether there are objects of which this conception is (even approximately) true. There is thus no need to presuppose the existence of natural numbers. Nevertheless, for stylistic reasons I will often write as if this presupposition is made.

Secondly, does our question presuppose that there are robust and general facts of the matter about what is responsible for our ability to single out individual natural numbers for

[9]This was observed in (Parsons, 1965) and discussed at length in (Wright, 1983). For a nice proof, see (Boolos, 1990).

reference? Could there not be different but equally legitimate conceptions of the natural numbers?[10] There could for instance be different conceptions of the natural numbers in cultures with no formal education, among educated lay people, and among professional mathematicians. I don't find this scenario particularly threatening. Even if it was true, our question could still be investigated provided it was suitably relativized to the community in question.[11] More thoroughgoing skeptics may worry that even relativized to a community, our question may not make sense or admit of a unique answer. I admit that this is a possibility and will accordingly not assume that this skeptical worry is unfounded. My response to this worry is more pragmatic. Let's see how much sense can be made of the question and, in light of this, to what the extent it admits of a unique answer.

4 Against the cardinal conception

I grant that the cardinal conception provides one possible way of thinking and talking about the natural numbers. But I deny that this is how we actually single out the natural numbers for reference in our most basic arithmetical thought and reasoning. I will now present some objections to the cardinal conception as an account of our actual arithmetical practice. The objections will be presented roughly in the order of increasing strength.

It should be noted that my concern in this section is quite different from that of some leading advocates of the cardinal conception. For instance, Hale and Wright are not concerned that their analysis should match our actual arithmetical practice. Their goal is rather to establish a possible route to *a priori* knowledge of an abstract realm of natural numbers.[12] This is clearly a legitimate and interesting goal. But in my view the more pressing question is to what extent actual arithmetical practice provides such knowledge. In particular, how and to what extent do ordinary educated lay people achieve such knowledge?[13]

[10]See also Alexander Paseau's contribution to this volume, where it is argued that our conception of the natural numbers is compatible with a reduction of arithmetic to set theory.

[11]My choice would then be to relativize the question to educated lay people from contemporary Western culture.

[12]See e.g. (Hale and Wright, 2000).

[13]This alternative goal is defended in (Heck, 2000) and (Linnebo, 2004), p. 168.

4.1 The objection from special numbers

This objection seeks to show that, if the cardinal conception had been correct, then certain special numbers would have been obvious and unproblematic in a way that they are not.

One such number is zero. It takes very little sophistication to know that certain concepts have no instances. For example, the concept of being a lump of gold in my pocket has no instances. So if our fundamental conception of natural numbers had the form $\#F$, then zero should have been a very obvious number. By contrast, the view that our most fundamental conception of the natural numbers is as ordinals predicts that zero should be just as non-obvious and problematic as the negative numbers; for every sequence of numerals has a first but no zeroth element. As it turns out, it was only at a very late stage in the history of mathematics that zero was admitted into mathematics as a number in good standing.[14] This suggests that our most fundamental conception of the natural numbers is ordinal-based rather than cardinal-based.

How convincing is this objection? I find it quite convincing against the view that our most fundamental conception of a natural number has the form $\#F$ where F is a concept. But the cardinal-based approach can perhaps be modified so as to block the objection. One may for instance take the most fundamental conception of a natural number to have the form $\#xx$ where xx is a plurality of objects. Since there are no empty pluralities, this would block the easy route to the number zero. However, this response would be a significant deviation from the traditional Frege-Russell view. And more seriously, it would undermine Frege's impressive bootstrapping argument for the existence of infinitely many natural numbers. For this argument makes essential use of the number zero.[15]

The objection from special numbers can also be developed for various infinite cardinals. If our fundamental conception of a natural number had been of the form $\#F$, then infinite cardinals should have been much more obvious and natural than they in fact were. For instance, not much sophistication is needed to grasp the concept of self-identity. And with a sufficient grasp of arithmetic comes a grasp of the concept of being a natural number. The cardinal conception therefore predicts that the numbers that apply to these two concepts

[14] Fibonacci is often credited with the introduction into the European tradition of an explicit numeral for zero in the early 13th century. But apparently even he did not regard zero as a proper number. According to the three-volume history of mathematics (Kline, 1972), "By 1500 or so, zero was accepted as a number" (p. 251).

[15] See (Shapiro and Weir, 2000) for a discussion.

should have been fairly obvious. But the history of mathematics shows that it took the great creative genius of Cantor to accept and explore the idea of infinite cardinal numbers. And when he did so, he met with wide-spread incomprehension and opposition. This historical evidence is better explained by the view that our fundamental conception is ordinal-based.

A natural response to this objection would be to modify HP so as to assign numbers only to finite concepts.[16] Let $\textsc{Fin}(F)$ be some formalization of the claim that the concept F has only finitely many instances. Then let *Finite Hume* be the following principle:

$$\text{(FHP)} \qquad \textsc{Fin}(F) \wedge \textsc{Fin}(G) \rightarrow (\#F = \#G \leftrightarrow F \approx G)$$

This modification clearly blocks the present argument. However, this way of blocking the argument is problematic. For as will be argued in Section 5.1, it is implausible that people with ordinary arithmetical competence should grasp, however implicitly, the concept of finitude.

4.2 The objection from the philosophy of language

This objection begins with the claim that in natural language, expressions of the form 'the number of Fs' are definite descriptions rather than genuine singular terms. This claim enjoys strong linguistic evidence. Certainly the surface structure of the expression indicates that it is a definite description rather than a genuine singular term. Moreover, expressions of this form are easily seen to be non-rigid. For instance, the number of bicycles in my possession is 4; so had I bought another, the number of bicycles in my possession would have been 5. By contrast, numerals are genuine singular terms and rigid designators.

The objection continues by claiming that our most fundamental conception of an entire category of objects cannot be based entirely on definite descriptions but must also involve some more direct form of reference. For in order to understand a definite description, one must be capable of some more direct way of referring to the objects in question. Consider for instance the sentence 'The tallest person in this room is male'. In order to understand this sentence one needs an ability to identify people which is prior to and independent of what is provided by definite descriptions. More generally, if F is a sortal and Φ a predicate defined on Fs, then in order to understand 'the F is Φ' one needs some more direct way of referring

[16] See (Heck, 1997).

to Fs than is provided by definite descriptions.[17]

The relevant contrast between numerals and descriptions of the form 'the number of Fs' has to do with their internal semantic articulation. My claim is that the descriptions cannot serve as a fundamental mode of reference to numbers because they have an internal semantic articulation which presupposes some more basic form of reference to numbers. By contrast, numerals are semantically simple expressions with no internal semantic articulation. This does not mean that their reference is primitive and unanalyzable; indeed I have argued that their reference admits of an analysis in terms of criteria of identity. But this is an account of what the reference of a semantically simple expression consists in, not an account of how the semantic value of a complex expression (such as a definite description) is determined by the semantic values of its simple constituents.[18]

4.3 The objection from lack of directness

If there are five apples on a table, we can think of the number five as the number of apples on the table. Or, following Frege, we can think of five as the number of (cardinal) numbers less than or equal to four. But neither or these ways of thinking of the number five feels particularly direct or explicit. Indeed, many people with basic arithmetical competence won't find it immediately obvious that the number of numbers between 0 and 4 (inclusive) is 5 as opposed to 4.

Rather, the only perfectly direct and explicit way of specifying a number seems to be by means of some standard numeral in a system of numerals with which we are familiar. Since the numerals are classified in accordance with their ordinal properties, this suggests that the ordinal conception of the natural numbers is more fundamental than the cardinal one.[19]

It may be objected that, since we don't have transparent access to all features of our thought, we cannot take at face value the kind of phenomenological evidence that I have just adduced about what kinds of reference *feels* most direct. This is a perfectly legitimate concern. But although the above considerations are particularly powerful when presented from a first-person point of view, there is no need to present them in that way. I conjecture that the results would be confirmed by a more objective, third-personal investigation of cognitive

[17] For a more developed argument of this sort, see (Evans, 1982), esp. Section 4.4.

[18] In the terminology of (Stalnaker, 1997) (which will be explained in Section 6), the former account belongs to *meta-semantics*, whereas the latter belongs to *semantics proper*.

[19] A view of this sort is defended in (Kripke,).

processing of arithmetical claims, for instance in terms of reaction times.[20]

4.4 Alleged advantages of the cardinal conception

I end this section by briefly considering two alleged advantages of the cardinal conception.

One alleged advantage is that the applications of the natural numbers are built directly into their identity conditions. This is just an instance of a more general requirement sometimes known as *Frege's constraint*.[21] But should this constraint be respected? We obviously need *some* account of how mathematics is applied. But why should the account have to build the applications of mathematical objects directly into their identity conditions?[22] Besides, even if Frege's constraint could be defended, this would not obviously favor the cardinal conception. For the natural numbers lend themselves to ordinal as well as cardinal applications, and Frege's constraint does not settle which of these applications should be built into the identity conditions of numbers.

Another alleged advantage of the cardinal conception is that it allows for Frege's famous "bootstrapping argument" for the principle that every number has an immediate successor. Mathematically this argument is extremely elegant and interesting. But as an account of people's actual arithmetical reasoning or competence it is implausible. The argument was developed only in the 1880s and is complicated enough to require even trained mathematicians to engage in some serious thought. So this is unlikely to be the source of ordinary people's conviction that every number has an immediate successor.[23]

5 Developing the ordinal conception

I now develop the ordinal conception of natural numbers in more detail.[24]

5.1 Refining the criterion of identity

Arithmetic teaches us that the natural numbers are *notation independent* in the sense that they can be denoted by different systems of numerals. In fact, an awareness of this notation

[20]Indeed, most cognitive psychologists appear to think that our capacity for exact representations of numbers (other than very small ones) is based on our understanding of some system of numerals.

[21]See (Wright, 2000) for a discussion and partial defense of this constraint.

[22]See also (Parsons, 2008), Section 14 for criticism of Frege's constraint.

[23]See (Linnebo, 2004), pp. 168-9.

[24]For a broadly similar account, see (Parsons, 1971), Section 3.

independence is implicit already in basic arithmetical competence. Even people with very rudimentary knowledge of arithmetic know that the natural numbers can be denoted by ordinary decimal numerals, by their counterparts in written and spoken English and other natural languages, and by sequences of strokes. Many people also know alternative systems of numerals such as the Roman numerals and the numerals of position systems with bases other than ten, such as binary and hexadecimal numerals. To accommodate the notation independence of the natural numbers, we need an account of acceptable numerals and a condition for two numerals to denote the same number.

I will take a numeral to be anything that can stand in a suitable ordering. On this very liberal view, a numeral need not even be a syntactical object in any traditional sense. For instance, if a pre-historic shepherd counts his sheep by matching them with cuts in a stick, then these cuts count as numerals. The purpose of a numeral is just to mark a place in an ordering. But of course, one and the same object can inhabit different positions in different orderings. The syntactical string 'III' can for instance mean either 3 or 111 depending on the ordering in which it is placed. We should therefore make explicit the ordering in which a numeral is placed. So I will take a numeral to be an ordered pair $\langle u, R \rangle$, where u is the numeral proper and R is some ordering in which u occupies a position.

How should the ordering R be understood? One question is whether R should be understood in an extensional or intensional manner. I see two reasons to favor the latter option. Firstly, in order to be learnable and effectively useable, the ordering must be computable. This means that the ordering must be understood as some sort of procedure rather than as an infinite collection of ordered pairs. Secondly, there are only finitely many numeral tokens of any learnable and effectively useable numeral system. This means that the ordering must be defined not only on actual numerals but in a way that extends to possible further numerals as well. This is best accounted for by considering orderings in intension.

Another question is about the order type of the relation R. A minimal requirement is that R be a discrete linear ordering with an initial object. For we want it to be the case that no matter how far we have counted, there is a unique next numeral—provided there are further numerals at all. The question is whether we have any reason to go beyond this minimal requirement.

We could for instance add the further requirement that R be of order type ω.[25] But I don't think that would be a good idea. The notion of being of order type ω is conceptually quite demanding. This notion was not clearly grasped until rather late in the history of mathematics, at the earliest in the late 16th century when mathematical induction was first explicitly articulated as one of the core principles of arithmetic. In fact, the problems that many people have with the idea of mathematical induction suggests that even an implicit grasp of the notion goes beyond what is required for basic arithmetical competence. This makes it doubtful that the notion should play a central role in people's most basic conception of the natural numbers.

What about the weaker requirement that R be without an end point? This requirement too seems undesirable. For it seems plausible to allow numeral systems based on finite orderings to denote natural numbers. (This may actually be the case on one construal of the Roman numerals.)

Finally, what about the requirement that R be well-founded or that its order type not exceed ω? These requirements too seem too demanding to be implicit in basic arithmetical competence. The fact that all traditional numeral systems are well-founded and of order type at most ω is explained better and much more simply by observing that these are the kinds of orderings with which we are familiar from the recursive formation rules of natural language.

I conclude that no requirement needs to be imposed on the order type of R other than the minimal one that R be a discrete linear ordering with an initial object. On this view, the natural numbers are clearly distinguished from non-standard numbers only when mathematical induction is introduced as a basic arithmetical principle.

I turn now to the equivalence relation that must hold between two numerals $\langle u, R\rangle$ and $\langle u', R'\rangle$ for them to determine the same number. This equivalence relation must clearly be a matter of the two objects u and u' occupying analogous positions in their respective orderings. More formally, $\langle u, R\rangle$ and $\langle u', R'\rangle$ are equivalent just in case there is a relation C which is an order-preserving correlation of initial segments of R and R' such that $C(u, u')$. I write $\langle u, R\rangle \sim \langle u', R'\rangle$ to symbolize that the two ordered pairs are equivalent in this sense.[26]

[25] Mathematical structuralists are often attracted to this requirement. See the works cited in footnote 3.

[26] Note that $\sim$ is guaranteed to be one-to-one only on finite numerals. This means that the "numbers" corresponding to any infinite numerals cannot be identified with ordinal numbers in the contemporary sense. Rather, these "numbers" are unintended and pathological objects which will (as described below) be ruled out as soon as one's conception of natural number is sophisticated enough to include the principle of induction.

A slight refinement of our previous two-level criterion of identity for natural numbers can now be expressed as follows:

(2L-$\mathbb{N}'$)
$$N\langle u, R\rangle = N\langle u', R'\rangle \leftrightarrow \langle u, R\rangle \sim \langle u', R'\rangle$$

where the variables R and R' are tacitly restricted to relations of the sort characterized above. Let '$\textsc{Num}(x)$' be a predicate that holds of all and only the objects that can be presented in this way.

5.2 Justifying the axioms of Dedekind-Peano Arithmetic

I now show how this conception of the natural numbers, supplemented with some natural and conceptually very simple definitions, allows us to justify all the axioms of Dedekind-Peano Arithmetic (PA). I proceed in three stages. First I introduce three relations on numerals which correspond to the basic arithmetical relations of succession, addition, and multiplication. Then I show that these relations on numerals induce corresponding relations on the natural numbers, thus allowing us to define the non-logical primitives of the language of PA. Finally I show how we can justify the axioms of PA on this basis.

We can define a successor relation $S^{\#}$ on numerals by letting $S^{\#}(\langle u, R\rangle, \langle u', R'\rangle)$ just in case u' is the R'-successor of some v such that $\langle u, R\rangle \sim \langle v, R'\rangle$. We now wish to define relations on numerals $A^{\#}$ and $M^{\#}$ which correspond to the arithmetical relations of addition and multiplication on natural numbers. Let $I^{\#}(u)$ formalize the claim that the numeral x is first in its ordering. Following the practice of ordinary counting, we think of such initial numerals as representing the number 1. Then the two desired relations are defined by means of the following recursion equations (where the free variables are implicitly understood as ranging over numerals):

(A1)
$$I^{\#}(y) \rightarrow [A^{\#}(x, y, z) \leftrightarrow S^{\#}(x, z)]$$

(A2)
$$S^{\#}(y, y') \wedge A^{\#}(x, y, z) \rightarrow [A^{\#}(x, y', z') \leftrightarrow S^{\#}(z, z')]$$

(M1)
$$I^{\#}(y) \rightarrow [M^{\#}(x, y, z) \leftrightarrow x \sim z]$$

(M2)
$$S^{\#}(y, y') \wedge M^{\#}(x, y, z) \rightarrow [M^{\#}(x, y', z') \leftrightarrow A^{\#}(z, x, z')]$$

It is easily verified that the three relations $S^{\#}$, $A^{\#}$, and $M^{\#}$ do not distinguish between numerals that are equivalent under $\sim$. These relations therefore induce corresponding relations on the natural numbers themselves, defined by

(Def-S)	$S(N(x), N(y)) \leftrightarrow S^{\#}(x, y)$
(Def-A)	$A(N(x), N(y), N(z)) \leftrightarrow A^{\#}(x, y, z)$
(Def-M)	$M(N(x), N(y), N(z)) \leftrightarrow M^{\#}(x, y, z)$

Following the practice of ordinary counting, we let 1 be the first number. For instance, 1 may be presented as $N\langle \text{`1'}, D\rangle$, where D is the standard ordering of the decimal numerals.

We can now verify that the various axioms of PA hold. Some are quite straightforward.

(PA1) $\textsc{Num}(1)$

(PA2) $\textsc{Num}(x) \rightarrow \neg S(x, 1)$

(PA3) $S(x, y) \wedge S(x', y) \rightarrow x = x'$

(PA4) $S(x, y) \wedge S(x, y') \rightarrow y = y'$

(PA5) $A(x, 1, z) \leftrightarrow S(x, z)$

(PA6) $S(y, y') \wedge A(x, y, z) \rightarrow [A(x, y', z') \leftrightarrow S(z, z')]$

(PA7) $M(x, 1, z) \leftrightarrow x = z$

(PA8) $S(y, y') \wedge M(x, y, z) \rightarrow [M(x, y', z') \leftrightarrow A(z, x, z')]$

(PA1) is trivial. For (PA2), let x be any number. Then there is a numeral $\langle u, R\rangle$ such that $x = N(\langle u, R\rangle)$. If $S(x, 1)$, then u would precede the initial object of the relevant ordering, which is impossible. (PA3) follows from the observation that any two numerals that immediately precede a third are equivalent. (PA4) follows from the observation that any two numerals that immediately succeed a third are equivalent. (PA5) - (PA8) follow readily from (A1), (A2), (M1), and (M2).

It is less obvious how to justify the next axiom:

(PA9) $\forall x(\textsc{Num}(x) \rightarrow \exists y S(x, y))$

It would clearly suffice if we could assume that every numeral bears $S^\#$ to some further numeral. But this assumption is doubtful, at least as concerns actual numeral tokens. Consider instead the weaker modal claim that necessarily, for any numeral there could be some further numeral to which it bears $S^\#$:

$$\Box\forall\langle u,R\rangle\Diamond\exists\langle u',R'\rangle\ S^\#(\langle u,R\rangle,\langle u',R'\rangle) \tag{1}$$

This weaker claim is extremely plausible. For assume we are given a numeral $\langle u,R\rangle$. Then it is possible that there is an object u' distinct from u and all of its R-predecessors. Let R' be the result of adding the pair $\langle u,u'\rangle$ to the initial segment of R ending with u. Then $\langle u',R'\rangle$ is as desired. (PA9) follows from (1) and the claim that numbers exist necessarily if at all.[27]

Finally, we need to specify some condition of finitude with which to restrict the numbers such that we get all and only the natural numbers. I claim that this condition is simply that mathematical induction be valid of the natural numbers. That is, x is a natural number (in symbols: $\mathbb{N}(x)$) just in case the following open-ended schema holds:

$$\phi(1)\wedge\forall u\forall v[\phi(u)\wedge S(u,v)\to\phi(v)]\to\phi(x)$$

This allows us to justify the final axiom of PA, namely the induction scheme:

(PA10) $\quad \phi(1)\wedge\forall u\forall v[\phi(u)\wedge S(u,v)\to\phi(v)]\to\forall x(\mathbb{N}(x)\to\phi(x))$

Prior to this scheme, it is doubtful that people had uniquely singled out the natural number structure as opposed to the objects of some non-standard model of arithmetic.

6 The metaphysical status of natural numbers

Does the account of reference to the natural numbers that I have outlined tell us anything about their metaphysical status? I now show that the account opens for the possibility of a "reductionist" interpretation of arithmetical discourse. But I argue that this interpretation should not be seen as a vindication of nominalism but rather as a reductionist account of what reference to natural numbers consists in.

[27]The proof uses "the Brouwerian axiom" $p\to\Box\Diamond p$. Another strategy is to settle for a modal theory of arithmetic based on the principle that necessarily for any number there could be a larger number. One can then give a very natural interpretation of ordinary Peano Arithmetic in this modal theory by interpreting the ordinary arithmetical quantifiers $\forall n$ and $\exists n$ as respectively $\Box\forall n$ and $\Diamond\exists n$. See my (Linnebo,) for details.

It is useful to begin by reminding ourselves of an important feature of most non-mathematical predications. Consider for example the question whether a physical body has the property of being a solid sphere. This question cannot be answered on the basis of any one of the body's proper parts; rather, information will be needed about the entire body. And there is nothing unusual about this example. Whether a body has some property generally depends on all or many of its parts. So a body has most of its properties in an irreducible way, that is, in a way that isn't reflected in any property of any one of its proper parts. This means that physical bodies play an ineliminable role in making predications true: for the truth of such predications cannot be reduced to a matter of how the body is presented.

The natural numbers are very different in this regard. Consider the question whether a natural number n has some arithmetical property, say the property of being even. Unlike the case of the roundness of a physical body, any numeral $\langle u, R \rangle$ by means of which n is presented suffices to determine an answer to the question, based on whether or not the numeral proper u occurs in an even-numbered position in the ordering R. There is no need to examine other presentations of n (or the number itself, whatever exactly that would involve). In fact, this observation can be seen to generalize: for any arithmetical property P, the question whether n possesses P can be reduced to a question about any numeral by means of which n is presented. A natural number is in this way "impoverished" compared to the numerals that present it, as all of its properties are already implicitly contained in each of these numerals. This opens for the possibility of a form of reductionism about natural numbers, as questions about such objects can be reduced to questions about their presentations. It then becomes hard to resist the idea that the natural numbers are mere "shadows of syntax" (in the apt metaphor of (Wright, 1992), pp. 181-2).

Given this reductionism, does it still make sense to say that numerals refer to natural numbers? I believe this is best understood as the question whether it still make sense to ascribe semantic values to numerals. There is strong *prima facie* reason to do so. When we analyze English and the language of arithmetic, singular terms such as '5' and '1001' seem to function just like terms such as 'Alice' and 'Bob'. The default assumption is therefore that all these terms function in similar ways. Since singular terms such as 'Alice' and 'Bob' clearly have semantic values (namely the physical bodies that they refer to), this provides *prima facie* reason to think that arithmetical singular terms such as '5' and '1001' have semantic values as well.

It may be objected that this *prima facie* reason is overridden by our discovery that questions about natural numbers can be reduced to questions about the associated numerals. Since this reduction shows that it suffices to talk about the numerals themselves, there is no need to ascribe any sort of semantic values to numerals. But this objection is much too quick. For the reducibility of questions about numbers to questions about numerals allows of two different kinds of explanation, only one of which allows the objection to go through. To see this, we need to distinguish between *semantics* and what is sometimes called *meta-semantics*.[28] Semantics ascribes semantic values to expressions and studies how the semantic value of a complex expression depends on the semantic values of its various simple constituents. But the relation between a linguistic expression and its semantic value is never a primitive one but one that obtains in virtue of some other and more basic facts. Compare the relation of ownership, which also isn't a primitive one. When I bear the ownership relation to my bank account, this isn't a primitive fact but one that obtains in virtue of some other and more basic facts. Meta-semantics is the study of what it is in virtue of which expressions have semantic values.

The distinction between semantics and meta-semantics points to two different ways of understanding the reducibility of questions about numbers to questions about numerals. The first way, which is the one presupposed in the above objection, is a form of *semantic reductionism*, according to which numerals don't have semantic values but serve some alternative semantic purpose. This would yield a nominalist interpretation of the language of arithmetic. The second way of understanding the reducibility is as a form of *meta-semantic reductionism*, namely a reductive analysis of the relation that obtains between a numeral and its semantic value. On this second analysis, it is perfectly true to say that the numerals have numbers as their semantic values. The point is rather that this truth admits of a reductionist analysis.

I defend the second account elsewhere.[29] If correct, what consequences will this have for the question of mathematical platonism? If by 'mathematical platonism' we mean simply the view that there are true sentences some of whose semantic values are abstract, then my view is obviously a platonist one.[30] But given how lightweight the relevant semantic values are,

[28]See e.g. (Stalnaker, 1997).
[29]See e.g. (Linnebo, 2009). I hope to return to this issue in future work.
[30]See Agustín Rayo's contribution to this volume for another discussion of lightweight forms of platonism.

this may be more of a reason to sharpen one's definition of 'mathematical platonism' than for platonists to declare victory.[31]

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