Øystein Linnebo
University of Bristol
Draft of 26 May 2006; minor revisions 21 March 2007

The nature of mathematical objects

1. *Frege's argument for mathematical platonism*

Philosophers classify objects as either concrete or abstract. Roughly speaking, an object is *concrete* if it exists in space-time and is involved in causation. Otherwise the object is *abstract*.[1] Someone who believes that there exist abstract objects is said to be a *platonist*, and someone who denies this is called a *nominalist*.

On the face of it, platonism seems very far removed from the scientific world view that dominates our age. Nevertheless many philosophers and mathematicians believe that modern mathematics requires some form of platonism. The defense of mathematical platonism that is both most direct and has been most influential in the analytic tradition in philosophy derives from the German logician-philosopher Gottlob Frege (1848-1925).[2] I will therefore refer to it as *Frege's argument*. This argument is part of the background of any contemporary discussion of mathematical platonism.[3]

Frege's argument begins with the observation that the language of mathematics contains expressions that are supposed to refer to abstract mathematical objects such as numbers, functions, spaces, and geometrical figures. We see this already from a casual inspection of the language of mathematics: this language has its own stock of proper names that are supposed to denote mathematical objects (for instance '5' and 'π'), and it contains quantifier phrases that are supposed to to range over mathematical objects ('for any natural number *n*' and 'there is a real number *x*'). Frege's argument continues by claiming that a lot of mathematical statements are true. Evidence for this claim is that lots of such statements are asserted in complete earnest by everyone from lay people to expert mathematicians, and that such statements are employed in everything from quotidian reasoning to advanced science. Combining this second premise about truth with the first premise about the semantic purpose

---

[1] This distinction between abstract and concrete is different from the one used in mathematics, where it means something like the distinction between general and particular.

[2] See in particular Frege (1884).

[3] What I call "Frege's argument" abstracts from certain aspects of Frege's own defense of mathematical platonism; most importantly, from his view that arithmetic is reducible to logic. These aspects are in my opinion best seen as providing further support for and explanation of the premises of what I call "Frege's argument." By thus detaching Frege's argument from optional add-ons, we get an argument with very broad appeal. For instance, the so-called "indispensability argument" for mathematical platonism, deriving from W.V. Quine and Hilary Putnam, can be seen as just another way of supporting and explaining the premises of Frege's argument.

of various mathematical expressions, Frege's argument concludes that there exist mathematical objects. For a sentence containing expressions that are supposed to refer to a certain kind of object cannot be true unless there really exist objects of the kind in question.

The two premises of Frege's argument take at face value certain apparent features of mathematical language and mathematical practice. But as we all know, appearances can sometimes deceive. Both premises have therefore been challenged.

Consider the first premise that the language of mathematics is supposed to refer to mathematical objects. Here one may challenge the classification of certain mathematical expressions as singular terms and quantifiers. For instance, one may argue that the adjectival use of the numerals (as in "there are five apples") is more fundamental than the substantival use (as in "the number of apples is five"). Indeed, it is a commonplace that the surface structure of natural language can deceive. Consider for instance a sentence like 'Tom did it for John's sake'. Although at a superficial level the expression 'John's sake' appears to be analogous to the expression 'John's car', a more careful logico-linguistic analysis reveals that these two expressions function differently. Frege himself was acutely aware of the danger of being misled by the surface structure of natural language. For this reason, he gave a sustained defense of his claim that certain mathematical expressions function logically as singular terms and quantifiers. He did this by developing the logical analysis of the language of mathematics which is now standard in philosophy and in mathematical logic. On this analysis, the numerals function logically as singular terms, and what look like quantifiers do indeed function as such. If this logical analysis is acceptable, then so will be the first premise of Frege's argument.

The second premise of Frege's argument states that the theorems of mathematics for the most part are true. Why, one may wonder, should we accept this claim? Can't we just regard these theorems as part of a useful game or a convenient fiction, and in this way avoid assigning any truth-values to them? On this sort of view, mathematical theorems would play a role that is strictly internal to a game or a fiction, and we wouldn't in full earnest have to accept them as true. Again, Frege has a response.[4] Unless we accept the theorems of mathematics as true, Frege says, we won't be able to account for their applicability. On the formalist and fictionalist interpretations of mathematics it remains a mystery how mathematics can be applied. For if mathematics isn't true, why should an empirical statement deduced from a true empirical theory along with a body of mathematics itself be true? The conclusion of a valid argument is guaranteed to be true only if all the premises are true, or so one would think. So Frege's response has great force.[5]

[4] See in particular Frege (1903), Section 91 and the surrounding discussion.

[5] But his response isn't conclusive as it stands. For it is not ruled out that mathematical theorems have

Although this defense of the premises of Frege's argument is less than conclusive, it at least shows that they enjoy great plausibility. No wonder, then, that Frege's argument has so profoundly changed the nature of the debate about mathematical platonism. The argument identifies two premises that are eminently natural and plausible, and it shows that mathematical platonism follows from these two premises. Frege's argument therefore succeeds in shifting the burden of proof onto the nominalist. Since our starting point is to believe in these two premises, we now need a reason *not* to believe in mathematical platonism.

2. *Two challenges to mathematical platonism*

To examine such reasons, I will now discuss two challenges to mathematical platonism. The first challenge is that mathematical platonism appears to make mathematical knowledge impossible.[6] How can the human mind reach out to the platonist's universe of abstract mathematical objects? Any causal relation is obviously out of the question, given that abstract objects aren't involved in causation. How then can our mathematical beliefs be sensitive to truths about this universe of abstract objects? In fact, it seems that our mathematical beliefs are *completely insensitive* to such truths. For what people believe is determined by facts about their brains and their physical environments. So the causal processes that take place in the physical world would have produced in us precisely the same mathematical beliefs regardless of the universe of abstract objects! This abstract universe appears to contribute nothing to the fact that we believe as we do. Even if this mathematical universe had not existed at all, our mathematical beliefs would have been precisely the same. The contrast with knowledge of the physical world is stark. My belief that there is a computer in front of me is caused by there actually being a computer in front of me. Had there not been a computer in front of me, I would not have believed that there is one. But mathematical platonists place the subject matter of mathematics outside of space-time and deny that it is involved in causation, and in so doing they foreclose the possibility of any causal explanation of mathematical knowledge. It therefore looks like the platonist's conception of mathematics makes mathematical knowledge impossible.

It may be responded that it is illegitimate to require, as this challenge does, that mathematical beliefs depend on or be sensitive to mathematical truths. For to say that X depends on or is sensitive to Y is, at the very least, to say that X *co-varies* with Y. But this claim makes no sense when Y consists of *necessary* truths. For a necessary truth could not

---

some property *other than truth* which guarantees the truth of all empirical statements that are deduced from a true empirical theory along with these mathematical theorems. For instance, Hartry Field has argued that the *semantic conservativeness* of pure mathematics is such a property. See Field (1980) and (1989).

[6] The classical source is Benacerraf (1973), but see also Field (1989), chapters 1 and 7.

have been any other way. Since mathematical truths are traditionally taken to be necessary, it therefore makes no sense to ask how other truths depend on them or are sensitive to them.[7]

This response is correct as far as it goes. But our first challenge to mathematical platonism can be stated so as to avoid this problem. We arrive at our mathematical beliefs by undergoing certain processes and by following certain methods. We would therefore like an account of how the processes and methods by which we arrive at our mathematical beliefs are *relevant* to what these beliefs are about. These processes and methods must somehow be appropriate for finding out about this subject matter. For surely it isn't just an *accident* that beliefs arrived at in these ways tend to be true.[8]

Can this challenge be met? The challenge confronts mathematical platonists with the following exercise of "solving for the unknown." Hold fixed our assumption that there is such a thing as mathematical knowledge and that this knowledge has an explanation. Then mathematical objects must be such that the methods by which we arrive at our mathematical beliefs are conducive to finding out about such objects. It is far from obvious that this equation has a solution. Since we cannot causally interact with abstract mathematical objects, the model that is appropriate to empirical knowledge doesn't apply to pure mathematics. And it remains a wide open question whether an alternative model exists which *is* appropriate to pure mathematics.

The second challenge to mathematical platonism is concerned with "Occam's razor," which instructs us not to postulate entities beyond necessity. Other things being equal, we should prefer lean scientific theories to ones with excess fat. By and large, this seems to be good scientific methodology. But not in mathematics! For mathematical objects are cheap.[9] In general, if mathematical objects answering to some natural description *can* exist, then they *do*. Given that imaginary numbers can exist, we assume (following Cardano and others) that they do. Given that "ideal numbers" can exist, we assume (following Dedekind and others) that they do. Given that various large cardinal numbers can exist, we assume (following past and present set-theorists) that they do. This view of correct mathematical methodology is nicely expressed in the following passage from the inventor of modern infinitary set theory, Georg Cantor.

> Mathematics is in its development entirely free and is only bound in the self-evident respect that its concepts must both be consistent with each other and also stand in

[7] A response of this sort is developed in Lewis (1986), pp. 111-12.

[8] For more on the ideas of this paragraph, see my Linnebo (2006).

[9] At least according to the norms that are standard in contemporary mathematics. But even on more restrictive views on mathematical method, such as constructivism, there will be a contrast between pure mathematics and empirical science of the sort I am calling attention to.

> exact relationships, ordered by definitions, to those concepts which have previously been introduced and are already at hand and established.
>
> In particular, in the introduction of new numbers it is only obligated to give definitions of them which will bestow such a determinacy and, in certain circumstances, such a relationship to the older numbers that they can in any given instance be precisely distinguished. As soon as a number satisfies all these conditions it can and must be regarded in mathematics as existent and real. […] for the *essence* of *mathematics* lies precisely in its *freedom*. (Cantor (1883), p. 896)

Most extravagant of all branches of mathematics is set theory. For arguably, the guiding norm in set theory is to maximize, to postulate as many sets as possible, stopping just short of inconsistency.[10] But this norm is diametrically opposite to Occam's razor!

This challenge confronts mathematical platonists with another exercise of "solving for the unknown." Hold fixed that the extravagant methods of modern mathematics are suited to finding out about mathematical objects. What is it about mathematical objects that makes these methods appropriate? Again it is unclear whether the platonists' equation has a solution. For it seems to follow from our commonsense idea of an object that objects should not be postulated lightly.

3. *From objects to semantic values*

We have seen that different considerations pull in different directions. On the one hand, Frege's argument for mathematical platonism has great plausibility. On the other hand, mathematical platonism faces two very serious challenges. To make progress, let's consider a radical proposal. Perhaps the two challenges appear insuperable only because we are operating with a wrong model of what "mathematical objects" are. Our model of objecthood has been physical bodies. According to this model, objects are like sticks and stones, apples and oranges. They are nuggets of stuff, lumps of reality. Being abstract rather than concrete, mathematical objects obviously cannot be *entirely* like sticks and stones. But they are supposed to be pretty much like sticks and stones except for being "outside of space-time" and causally isolated from us. Now, if *this* is what mathematical objects are, how on earth can our mathematical methods be conducive to our finding out about them? And how can such utterly exotic things be postulated so lightly?

Can we find a better model of what it is to be an object? I believe progress can be made by returning to Frege's argument. The argument's first premise, we recall, says that the language of mathematics contains expressions that contribute to sentences in which they occur in a way that is similar to how more familiar proper names contribute to sentences in which *they* occur. The similarity is that both kinds of expressions refer to objects. But in our present context, this characterization of the similarity isn't very helpful, given that our goal is

---

[10] For documentation, see Maddy (1997), esp. p. 131.

precisely to explicate what it is to be an object. More helpful is the notion of a *semantic value*, which plays a fundamental role in contemporary semantics and philosophy of language. Very briefly, this notion can be explained as follows. Each component of a sentence makes some definite contribution to the truth or falsity of the sentence. This contribution is its *semantic value*. Consider a simple sentence such as 'George W. Bush is president'. The proper name 'George W. Bush' makes a definite contribution to the truth of this sentence, namely its referent, George W. Bush. Likewise, the predicate 'is president' makes a definite contribution, namely a specification of what is required of an object for the predicate to be true of it; in this case, that the object be president (as opposed to, say, being king). Moreover, the truth or falsity of a sentence is determined as a *function* of the semantic values of its constituents. This is known as *the principle of compositionality*. For instance, in our toy sentence, it doesn't matter how the semantic value George W. Bush is picked out. If the proper name 'Dubya' has the same semantic value as 'George W. Bush', then 'Dubya is president' must have the same semantic value (in this case, truth-value) as our toy sentence.

The second premise of Frege's argument is that many mathematical sentences are true. For this to be the case, all expressions involved in these sentences must succeed in making their appropriate semantic contributions. Using our new terminology, this means that all of these expressions must have semantic values, and that these semantic values must combine so as to make the relevant sentences true. The conclusion of Frege's argument can now be re-stated as the claim that mathematical singular terms have semantic values. As we have seen, these semantic values (unlike those of proper names such as 'George W. Bush') cannot be identified with any concrete objects. Perhaps we can shed light on mathematical objects by explaining how mathematical singular terms manage to have semantic values and what the nature of these semantic values is.

To develop this idea, we need a better understanding of what is involved in our reference to various sorts of objects. Since human beings are very complex organisms who stand in very complex social relations, we will have to simplify. What we want is a good model, not necessarily a description that is accurate in every detail. I therefore propose that we develop a model of our reference to various sorts of objects based on robots (or computers embedded in, and interacting with, a physical environment). I will thus investigate under what conditions it makes sense to ascribe to robots a semantics involving reference to different sorts of objects. I will focus on two fundamental cases: reference to physical bodies and reference to natural numbers.

4. *Reference to physical bodies*

What is required of a robot for it to refer to physical bodies, such as sticks and stones, in its environment? I believe we get a better understanding of the problem by focusing on the senses of sight and touch, and on some very fundamental thought processes. Other senses, such as smell, taste, and hearing, appear to play a less fundamental role in our reference to physical bodies. Consciousness too (in the sense of awareness of what it is like to have various sorts of experiences) will be put to one side, as it appears inessential to our core notion of reference.[11]

So consider a robot equipped with senses of sight and touch similar to our natural human senses. Such a robot must interact with its environment by detecting light reflected by surrounding surfaces and by having a capacity for touching and grasping things in its vicinity. What is required for such a robot to make reference to a physical body in its environment? Obviously the robot must "perceive" the body in the sense that it must receive light from some part of its surface or touch some part of it. The robot will thus receive information from some spatiotemporal part of the body. These parts need not have natural boundaries in either space or time; they are simply the sum-totals of the particle-instants with which the robot causally interacts in this perception-like way.

But it is not sufficient for a robot to refer to a body that it should receive such information from some part of the body. The robot also needs a mechanism for determining when two such pieces of information belong to *the same* body. This task is far from trivial. For we are surrounded by bodies that are partially hidden, that are occluded by other bodies, and that move in and out of view. For instance, a stick can be partially buried, and a stone can be partially covered by other stones piled up around it. So there will always be different ways of "getting at" one and the same physical body, both from different spatial points of view and at different moments of time. It is therefore essential that the robot have some mechanism for grouping together pieces of perceptual information that belong to one and the same body.

I claim that what matters for this task is that the parts from which the robot receives perceptual information be spatiotemporally connected in some suitable way (to be spelled out shortly). Assume for instance that the robot establishes visual contact with part of a stick that emerges from the ground and that one of its "arms" is simultaneously probing into the ground nearby and encountering something hard. What should we "teach" the robot about the conditions under which the two parts it interacts with belong to the same body? Roughly, the kind of connectedness that matters has to do with solidity and motion: The two parts must be related through a continuous stretch of solid[12] stuff, all of which belongs to the same unit of independent motion (roughly in the sense that, if you wiggle one part, the other part follows along).

---

[11] This section draws on my Linnebo (2005), especially Section 4.

[12] I here mean 'solid' in the ordinary sense in which a stick or a stone is said to be solid. Of course, physics tells us that even sticks and stones aren't solid in the stricter sense of filling up all space at an atomic level.

To produce a more precise answer, think of this as an exercise in robotics. I submit that the following fundamental principles are part of an analysis of the concept of a physical body, and will therefore have to be implemented in the robot.[13]

(B1) Bodies are three-dimensional, solid objects.

(This principle holds because any two parts of a three-dimensional solid are naturally connected in space.) Thus, a cloud of gas doesn't qualify as a body in the present sense. This means that not all spatiotemporal objects are *bodies*.

(B2) Bodies have natural and relatively well distinguished spatial boundaries.

For instance, an undetached half-rock fails to be a body because it lacks sufficiently natural boundaries, and a mountain fails because its boundaries are insufficiently well distinguished.[14]

(B3) Bodies are units of independent motion.

Thus, although a book is a body, a pile of papers is not.

(B4) Bodies move along continuous paths.

Consider the object that came into being with the birth of Bill Clinton, coincided with Clinton until the end of his presidency, and thenceforth coincides with George W. Bush. By (B4) this object cannot be a body.

(B5) Bodies have natural and relatively well distinguished temporal boundaries.

So arbitrary temporal parts of bodies are not themselves bodies.

I believe this relatively simple model captures the core of the phenomenon of reference to physical bodies. What matters is that our agents (whether human or robot)

---

[13] These principles are also constitutive of the concept of what psychologists sometimes call "Spelke-objects." This concept corresponds closely to my concept of a physical body. See e.g. Spelke (1993) and Xu (1997).

[14] Precisely how well distinguished must the boundaries of a body be? Presumably, a shedding cat is still a physical body despite all the hairs that are in the process of falling off. Although I doubt that our question allows of any precise answer, I am hopeful that an approximate answer can be given by empirical investigation of ordinary people's concept of a body.

receive sensory information from parts of bodies and that they have a capacity for grouping together such pieces of information just in case these pieces derive from parts that are spatiotemporally connected in the way spelled out above. Let ~ be this relation of spatiotemporal connectedness. This is an equivalence relation on parts of bodies. This equivalence relation determines a (partial) function B that maps a part u to the physical body, if any, that u picks out. That is, $B(u)$ is the body that u is part of. Bodies are then subject to the following criterion of identity:

(Id-B) $\forall u \forall v\ (B(u) = B(v) \leftrightarrow u \sim v)$

5. *Reference to natural numbers*

Might it be the case that the structure involved in reference to physical bodies is just an instance of a more general phenomenon? Perhaps reference *always* consists in some relation to parts or aspects of objects, accompanied by some mechanism for determining when two such parts or aspects pick out the same object.[15]

Let's attempt to apply this idea to the natural numbers. Instead of information causally linked to some part of a body, a natural number is presented by means of a numeral. In fact, the most immediate ways in which a natural number is presented to people from contemporary Western culture is by means of an ordinary decimal numeral. So assume that our robots too operate with this system of numerals. However, no system of numerals can be *identified with* the natural numbers. For it is part of ordinary arithmetical competence that the natural numbers are "notation independent," in the sense that they can be denoted by different systems of numerals. Even people with a very rudimentary knowledge of arithmetic know that the natural numbers can be denoted not only by ordinary decimal numerals but also by their counterparts in written and spoken English (and in other natural languages) and by sequences of strokes (perhaps grouped in fives). Many people also know alternative systems of numerals such as the Roman numerals and the numerals of position systems with bases other than ten, such as binary and hexadecimal numerals.

I will here take a numeral to be any object that occupies a position in a well-ordering. In fact, since it is convenient to make the well-ordering explicit, I will take a numeral to be an ordered pair $\langle u, R\rangle$, where u is the numeral proper and R the well-ordering in which u occupies

[15] This section and the next draw on my Linnebo (forthcoming), especially Sections 3 and 4. The most influential attempt to account for reference to natural numbers by means of equivalence relations on other entities is due to Wright (1983); see also Hale and Wright (2001), especially the Introduction. Their attempt differs from mine in two main respects. Firstly, they take natural numbers to be presented by means of *concepts*, and they take the equivalence relation on such presentations to be that of one concept's being in one-to-one correspondence with another. Secondly, they deny that their account brings with it any sort of reductionism, whereas I argue in the next section that mine does.

a position. On this very liberal view of the matter, the numeral proper u need not be a syntactical object, at least not in any traditional sense. (For instance, if a pre-historic shepherd counts his sheep by matching them with cuts in a stick, then these cuts count as numerals.) Moreover, since R can be any well-ordering, these numerals refer to ordinal numbers but not necessarily to finite ones.

Next, we need to equip our robot with a general condition for when two numerals determine the same number. A moment's reflection shows that two numerals $\langle u, R\rangle$ and $\langle u', R'\rangle$ determine the same number just in case u and u' occupy analogous positions in their respective orderings; for instance, that both occupy the 17th position. This can easily be given a precise mathematical definition and (at least in principle) implemented in our robot. Let $\approx$ be the resulting equivalence relation on numerals.[16] This equivalence relation determines a function N that maps a numeral to the number that it determines:

(Id-*N*) $$N\langle u, R\rangle = N\langle u', R'\rangle \leftrightarrow \langle u, R\rangle \approx \langle u', R'\rangle$$

The numbers to which the numerals are mapped are not equivalence classes of numerals but form their own category of objects. (Compare the physical bodies to which the function B from the previous section maps parts of physical stuff. These physical bodies are not equivalence classes of their parts but form a distinct category of objects.) And the criterion of identity (Id-*N*) tells us how these objects are identified, just like the criterion of identity (Id-*B*) tells us how physical bodies are identified. Let O (for "ordinal") be a predicate that holds of all and only objects in the range of N.

Next we want to define a relation $P^{\#}$ that holds between two numerals $\langle u, R\rangle$ and $\langle u', R'\rangle$ just in case the number determined by the former immediately precedes the number determined by the latter. One easily sees that the right definition is that u' has an R'-predecessor v such that $\langle u, R\rangle \approx \langle v, R'\rangle$. It is then easy to verify that $P^{\#}$ doesn't distinguish between numerals that are equivalent under $\approx$. This means that the relation $P^{\#}$ on numerals induces a predecessor relation P on the ordinals themselves, defined by

(Def-*P*) $$P(N\langle u, R\rangle, N\langle u', R'\rangle) \leftrightarrow P^{\#}(\langle u, R\rangle, \langle u', R'\rangle).$$

[16] The relation $\approx$ may be taken to be a linguistic object, not a properly mathematical one, by identifying it with a formula with two free variables in an interpreted language.

Finally, following the ordinary practice of counting, we let 1 be the first number. We may for instance define 1 as $N\langle$'1', $D\rangle$, where D is the familiar well-ordering of base 10 numerals for positive integers.

With these definitions, it is easy to establish some of the basic axioms for ordinal numbers, for instance:

(O1) $O(1)$

(O2) $\neg\exists x\, P(x, 1)$

(O3) $P(x, y) \wedge P(x', y) \rightarrow x = x'$

(O4) $P(x, y) \wedge P(x, y') \rightarrow y = y'$

For instance, (O3) follows from the fact that any two numerals that precede a third are equivalent under $\approx$, which means that they determine the same ordinal.

However, I have not yet said anything very substantial about *how many* ordinals there are. For the purpose of describing the natural numbers (which I identify with the finite ordinals), the only such principle we need is an axiom that ensures the existence of successors:

(O5) $\forall x\, (O(x) \rightarrow \exists y P(x, y))$

This axiom doesn't follow from what has been said so far about the numerals. But the axiom can be motivated as follows. Begin with the extremely plausible principle that for any numeral, there *could be* (roughly, that it is consistent that there is) another numeral directly succeeding it. By (Def-*P*), this means that for any ordinal, there *could be* another ordinal directly succeeding it. From this we get (O5) by invoking the principle that any ordinal that could exist, does exist.

Finally, we need to specify some condition of finitude with which to restrict the ordinals such that we get all and only the natural numbers but no infinite ordinals. I claim that this condition is simply that mathematical induction should be valid of the natural numbers. That is, an ordinal n is a *natural number* just in case the following schema holds for any predicate φ:

(MI) $\varphi(1) \wedge \forall x \forall y[\varphi(x) \wedge P(x, y) \rightarrow \varphi(y)] \rightarrow \varphi(n)$

Our characterization of the natural numbers has thus allowed us to derive all the familiar axioms of Peano Arithmetic.

6. *The "thinness" of the natural numbers*

I will now explain a fundamental difference between physical bodies and natural numbers that has to do with the ways in which these objects possess properties.

Consider the question whether a physical body *x* has some property, say being round. To answer this question, it isn't sufficient to consider any proper part of *x*. Whether a body is round isn't determined by any of its proper parts but information is needed about the entire body. And there is nothing unusual about this case. It is in general true that, in order to determine whether a body *x* has some property *G* (such as a particular shape or mass), one needs information about many or all parts of *x*. The question whether a body has some property *G* cannot in general be reduced to a question about any *one* of its proper parts. This means that a body can have properties in an irreducible way, that is, in a way that isn't reflected in any property of any *one* of its proper parts.[17]

The situation is very different with the natural numbers. Consider the question whether a natural number *n* has some arithmetical property *G*, say the property of being even. In this case, unlike the situation of roundness of physical bodies, a standard presentation of *n* by some numeral (say a standard decimal numeral $\underline{n}$) suffices to answer the question. There is no need to examine other presentations of the number *n* or the number itself. In fact, the question whether the natural number *n* possesses an arithmetical property *G* can always be reduced to a question about the numeral $\underline{n}$ by which the number *n* is presented. For all the usual arithmetical properties are definable (in second-order logic) from the predecessor relation *P*. And as (Def-*P*) shows, the question whether *P* holds between two natural numbers is itself reducible to the question whether the relation $P^{\#}$ holds between certain numerals.

This means that on the view I have defended, the natural numbers are "impoverished" compared to numerals. For whenever a natural number *n* possesses some arithmetical property, its doing so is inherited from the fact that the numerals that present *n* possess some related property. Natural numbers are therefore "thinner" than the numerals that present them. In fact, since questions about natural numbers can be reduced to questions about the numerals that present them, this opens for a form of reductionism about natural numbers.

Given the possibility of this reductionism, does it still make sense to say that numerals *refer to natural numbers*? In light of Section 3, this question is best understood as asking whether it still makes sense to ascribe *semantic values* to numerals. I will now argue that this *does* still make sense. One observation that supports this claim is the following. The default assumption is that expressions that belong to the same syntactic category—in this case, the class of singular terms—should belong to the same semantic category as well. And

[17] The converse is true as well (although less important to the present discussion): a proper part can have properties that aren't reflected in any properties of the body of which it forms a part.

indeed, when we analyze English and the language of arithmetic, singular terms such as ‘5’ and ‘1001’ seem to function just like singular terms such as ‘Alice’ and ‘Bob’. But it is uncontroversial that singular terms such as ‘Alice’ and ‘Bob’ have semantic values, namely the physical bodies that they refer to. This provides at least some reason to think that arithmetical singular terms such as ‘5’ and ‘1001’ have semantic values as well.

It may be objected that this default assumption is overridden by our discovery that questions about natural numbers can be reduced to questions about the associated numerals. Since this reduction shows that it suffices to talk about the numerals, the objection continues, there is no need to ascribe any sort of semantic values to numerals. However, this objection assumes *that the structure responsible for the reduction that we have discovered is also the kind of structure that matters for semantic analysis*. I argue in other work that this condition is not met and that the objection therefore fails.[18] Taking a broader look at the issue, this shouldn’t be very surprising. For not every kind of structure that is involved in the phenomenon of reference is semantic structure. For instance, reference is often based on perception, and perception is undoubtedly a complicated process that involves all kinds of structuring of sensory information. But this structure will typically not be semantic structure. Although perception is often *presupposed by* the relation of reference and thus also by semantics, perception and its structure aren’t thereby *included in* semantics.

If I am right about that the objection fails, then it still makes sense to ascribe semantic values to numerals. And since these semantic values are nothing other than the natural numbers, this means that the numerals do after all refer to natural numbers.

7. *Back to the two challenges*

I have argued that natural numbers are fundamentally different from physical bodies by being so “thin.” Can this be used to answer the two challenges discussed in Section 2? I proceed in reverse order.

The second challenge was to explain why it is reasonable to operate with such “extravagant” methods as those of modern mathematics rather than the more “parsimonious” methods found in the empirical sciences. What is it about mathematical objects that makes it appropriate to postulate such objects so much more lightly than we postulate physical objects? When we translate talk about objects into talk about semantic values, the question becomes why so much less is required for a mathematical singular term to have a semantic value than for a physical singular term to have one. This is a question that we are now well equipped to answer. If pure mathematics is at all like arithmetic, then very little is required for a mathematical singular term to have a semantic value. All that is required is that the term be

[18] See Linnebo (forthcoming), Section 4.

associated with some (possibly syntactic) object that serves as a presentation of some semantic value, and that we have some principled and well-founded way of telling when two such presentations determine the same semantic value. This supports a view of mathematics like the one expressed in the passage from Cantor quoted in Section 2.

The first challenge was to explain how knowledge of mathematical objects is possible without any causal interaction with such objects. How can the methods by which we arrive at our mathematical beliefs be sensitive to the facts that are involved in making these beliefs true? Let's begin by considering a simple belief about the physical world such as 'This body is round' (pointing at a near-perfect globe). The truth of this belief[19] depends on two things: first, that the belief has some particular proposition as its semantic content, and second, that the world is such as to make this proposition true. I will now describe these two kinds of dependence and explain how both contribute to the formation of true beliefs.

Let's begin with the first kind of dependence. Why does the belief have this particular proposition as its content? On the account I have been developing, this question can be reduced to the question why the various *simple constituents* of the belief have the semantic values that they happen to have. This is a question about which I have had quite a lot to say. I have proposed a model of how the expression 'this body' comes to refer to a particular body, in this case the globe. This involves facts about the causal transmission of information from the globe to our sense organs and about this information's being put together in a way that is sensitive to the natural spatiotemporal connectedness of the chunks of physical stuff from which it derives.[20] Since these facts make it the case that the belief has some particular proposition as its content, they contribute *semantically* to the truth of the belief in question.

The second kind of dependence requires less comment. Since the content of the belief in question is the proposition that the relevant globe is round, this globe's actually being round obviously contributes to making the belief true. Now, the fact that the globe is round isn't among the facts that contribute semantically to the truth of the belief in question. I will therefore say that it contributes *non-semantically*. Note that we are not distinguishing between two kinds of facts—the semantic and the non-semantic—but rather between two kinds of *contribution* that a unique realm of facts can make to the truth of a belief.

Facts that contribute to making a belief true in either of these ways typically also contribute to an agent's formation of this belief. Let's begin with the facts that contribute non-semantically—in our example, the globe's actually being round. This fact obviously contributes to the agent's formation of the belief in question. Had this globe been seriously

[19] Here and in what follows I use the word 'belief' to mean a particular sort of internal psychological state, considered in abstraction from any propositional content that this state may have. This is thus a syntactic, rather than semantic, notion of belief.

[20] A story can also be told about how 'is round' comes to have *its* semantic value. This story will crucially involve the fact that our subject takes this predicate to apply to all and only round things.

dented, say, the agent would have noticed and therefore not formed the belief in question. What about the facts that contribute semantically? Recall that these are facts about the agent's being in perceptual contact with the globe and about the resulting perceptual information's being put together in accordance with the principle (Id-*B*) (see the end of section 4). These facts too contribute to the formation of the belief. Had the agent been in perceptual contact with another body, or had he put together pieces of perceptual information in accordance with some principle other than (Id-*B*), he would most likely not have formed the belief in question.[21] So in this example, a complete account of why the agent formed the belief in question must appeal both to the facts that contribute semantically to the truth of the belief and to the fact that contribute non-semantically.

I turn now to a very simple mathematical example, involving the mathematical belief that 2 directly precedes 3. (Once this example is in place, more complex examples can be given by exploiting the fact that other arithmetical relations are definable from the predecessor relation.) In this example too there are facts that ensure that the constituents of the belief have the semantic values that they happen to have. First there is the fact that the numerals '2' and '3' occupy the second and third positions of the standard sequence of decimal numerals. Then there is the fact that the agent takes two numerals to determine the same number just in case they stand in the equivalence relation $\approx$. Finally there is the fact that the agent takes the predecessor relation P's holding of two natural numbers to be a matter of the associated numerals' standing in the relation $P^{\#}$, as described in (Def-*P*). Unlike the previous example, however, there are no facts whose contribution to the truth of the belief that 2 directly precedes 3 is completely non-semantic. This kind of contribution to truth has vanished entirely.[22]

Fortunately, the facts that contribute semantically to the truth of the belief that 2 directly precedes 3 also suffice to explain why an agent formed this belief. Because the agent treats the predicate 'directly precedes' in accordance with (Def-*P*), he regards the belief that 2 directly precedes 3 as true just in case the associated numerals stand in the relation $P^{\#}$. And because he regards the associated numerals '2' and '3' as ordinary decimal numerals, he deems that they indeed stand in the relation $P^{\#}$. Consequently he regards the belief as true. Had the agent not treated the predicate 'directly precedes' in accordance with the definition

[21] Much the same goes for the facts involved in giving the predicate 'is round' its semantic value.

[22] Whether or not this makes the belief in question analytic will depend on how one understands the notion of analyticity. I would deny that the belief is analytic in the traditional sense that anyone who grasps the belief can see it to be true by conceptual analysis alone. For the semantic facts that I have been talking about need not be consciously accessible even to people with a perfect grasp of the belief. (Similarly, the semantic facts involved in reference to physical bodies need not be consciously accessible even to people who are fully competent with such reference.)

(Def-*P*), or had he not taken the numerals '2' and '3' to be ordered as decimal numerals, he would not have formed the belief.

Summing up, it turns out that the very same facts that make our sample mathematical belief true are also responsible for making the agent form the belief. The agent's belief is therefore appropriately sensitive to the truth of the belief, which answers the first challenge.

8. *Conclusion*

I began by outlining two conflicting views on mathematics: first Frege's argument that there exist abstract mathematical objects, and then two serious challenges to the idea that there could exist such objects. To make progress, I suggested that we must reject the standard conception of objecthood (which is modeled on the notion of a physical body) and instead use the technical notion of a *semantic value* to explicate the notion of objecthood. I then gave an account of reference to physical bodies, based on the idea that we perceptually interact with parts of such bodies and that we operate with an equivalence relation which tells us when two such parts belong to the same body. I next suggested that a similar account is possible of reference to natural numbers: natural numbers are presented to us by means of numerals, and we operate with an equivalence relation that tells us when two such numerals determine the same number. Natural numbers are on this view much "thinner" than physical bodies, in the sense that all properties of a natural number can be reduced to properties of the corresponding numeral, whereas not all properties of a physical body can be reduced to properties of its individual parts. I finally observed that on this conception of numbers (and of mathematical objects more generally) as "thin," we are able to *both* agree with Frege's argument *and* answer the two challenges to which this argument gives rise.

This means that we may continue to use platonistic language when thinking, talking and teaching about mathematics. For there is a perfectly legitimate sense in which we succeed in referring to mathematical objects. And there is nothing scientifically suspect about this form of platonism—at least not when the mathematical objects are understood as "thin."[23]

References

Benacerraf, Paul. 1973. "Mathematical Truth," repr. in P. Benacerraf and H. Putnam (eds.). 1983. *Philosophy of Mathematics: Selected Readings* (Cambridge: Cambridge University Press)

Cantor, Georg. 1883. *Foundations of a General Theory of Manifolds*. Transl. and repr. in William Ewald (ed.), *From Kant to Hilbert*, two volumes (Oxford: Clarendon, 1996)

[23] Thanks to Matti Eklund, Frode Kjosavik, and the editors of this volume for very valuable comments on an earlier version.

Field, Hartry. 1980. *Science without Numbers: A Defense of Nominalism* (Princeton, NJ: Princeton University Press)

---------------. 1989. *Realism, Mathematics, and Modality* (Oxford: Blackwell)

Frege, Gottlob. 1884. *Grundlagen der Arithmetik*. Transl. by J.L. Austin as *Foundations of Arithmetic* (Oxford: Blackwell, 1953)

-----------------. 1903. *Grundgesetze der Arithmetik*, Vol. 2. Excerpts transl. in P. Geach and M. Black (eds.), *Translations from the Philosophical Writings of Gottlob Frege* (Oxford: Blackwell, 1952), pp. 139-224

Hale, Bob and Crispin Wright. 2001. *The Reason's Proper Study* (Oxford: Clarendon)

Linnebo, Øystein. 2005. "To Be Is to Be an *F*," *Dialectica* 59(2), pp. 201-222

---------------------. 2006. "Epistemological Challenges to Mathematical Platonism," *Philosophical Studies* 129(3), pp. 545-574

---------------------. Forthcoming. "Frege's Context Principle and Reference to Natural Numbers," in Sten Lindström (ed.), *Logicism, Formalism, Intuitionism: What Has Become of Them?* (New York: Springer)

Maddy, Penelope. 1997. *Naturalism in Mathematics* (Oxford: Clarendon)

Spelke, Elizabeth. 1993. "Object Perception," in A. Goldman (ed.), *Readings in Philosophy and Cognitive Science* (Cambridge, MA: MIT Press)

Wright, Crispin. 1983. *Frege's Conception of Numbers as Objects* (Aberdeen: Aberdeen University Press)

Xu, Fei. 1997. "From Lot's Wife to a Pillar of Salt: Evidence that *Physical Object* is a Sortal Concept," *Mind and Language* 12, pp. 365-92