Pluralities and Sets*

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1 When do some things form a set?

Say that some things *form a set* just in case there is a set whose members are precisely the things in question. For instance, all the inhabitants of New York form a set. So do all the stars in the universe. And so do all the natural numbers. Under what conditions do some things form a set?

Let us make the question more precise. Plural expressions such as 'some things' are best explained by means of *plural logic*. In ordinary singular first-order logic we have singular variables such as x and y, which can be bound by the existential and universal quantifiers. In *plural first-order logic* we have in addition plural variables such as xx and yy, which can also be bound by existential and universal quantifiers to yield so-called *plural quantifiers*.[1] '$\exists xx$' is read as 'there are some things xx such that ...', and '$\forall xx$', as 'given any things xx, ...'. These quantifiers are subject to inference rules analogous to those of the ordinary singular quantifiers. There is also a two-place logical predicate $\prec$, where '$u \prec xx$' is to be read as 'u is one of xx'.

By 'set' I mean set as on the standard iterative conception, according to which the sets are "formed" in stages.[2] We begin at stage 0 with all the non-sets, which are known as

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[1] Plural first-order logic was made popular by George Boolos, "To Be Is to Be a Value of a Variable (or to Be Some Values of Some Variables)," this Journal, LXXXI (1984): 430-439. For an introduction, see Øystein Linnebo, "Plural Quantification," in Edward N. Zalta, ed., *Stanford Encyclopedia of Philosophy* (2008).

[2] See George Boolos, "The Iterative Conception of Set," this Journal, LXVIII (1971):215-32. It is of course controversial how the metaphor of "set formation" should be understood. I will provide a modal explication in Section 5, following Charles Parsons, "What Is the Iterative Conception of Set?," repr. in his *Mathematics in Philosophy* (Ithaca: Cornell, 1983), pp. 268-297.

urelemente. At stage 1 we form all sets of objects available at stage 0. We then continue in this way, at each stage forming all the sets that can be formed from objects available at the preceding stages.[3] What if there are other defensible conceptions of sets? Since I will here only be concerned with sets on the standard iterative conception, I need not take a stand on this. If need be, all of my quantifiers can be regarded as implicitly restricted to the standard iterative hierarchy of sets (based on *urelemente*).

Say that some things *xx form a set y* just in case $\forall u(u \prec xx \leftrightarrow u \in y)$, and let $\text{Form}(xx, y)$ abbreviate this claim. I can now give a precise expression to the question with which I began. To ask under what conditions some things form a set is to ask what is required of some things xx for there to be a set y such that $\text{Form}(xx, y)$.

Since this is a question about what sets there are, a natural thought is simply to consult standard set theory. However, the usual axiomatizations of set theory, such as ZFC, are formulated in a language without any plural vocabulary. So in this language our question cannot even be properly expressed. The best we could hope for is that standard set theory will answer our question in an indirect way. Standard set theory places substantive constraints on what sets there can be; for instance, it disallows a universal set. Perhaps these constraints suffice to determine what is required of some things for them to form a set. However, it will turn out that this hope is quashed, because the constraints in question are compatible with different accounts of what pluralities form sets. The underlying problem with the strategy of simply consulting standard set theory is that our question is not only about what sets there are, but equally about how plural logic interacts with set theory. When addressing this question we must obviously respect the claims of ordinary singular set theory. But we must also go beyond these claims and analyze the concepts of plurality and set.

By far the simplest answer to our question is that nothing is required for some things to form a set. According to this answer, there is a collapse of pluralities to sets:[4]

(Collapse) $$\forall xx \exists y \text{Form}(xx, y)$$

Note that the claim is not that plural variables range over sets. The claim is the weaker one

[3]More precisely, we start at stage 0 with some urelemente, whose set V_0 we then form. If this process has been carried out up to some ordinal α, the next step is to let $V_{\alpha+1} = \wp(V_\alpha)$. If the process has been carried out for all ordinals less than some limit ordinal λ, we let $V_\lambda = \bigcup_{\gamma<\lambda} V_\gamma$.

[4]It is often convenient to use the locution 'plurality' when talking about some things. This is intended to be neutral on the question whether plural logic is ontologically committed to plural entities of some sort. (This neutrality will be maintained throughout the article.) Readers may translate claims involving the locution 'plurality' back into the official idiom provided by plural logic.

that given any objects, there is a set based on precisely these objects.

The view expressed by COLLAPSE is not only simple but has great intrinsic plausibility. If it were not for some problems to be described below, we would probably all have believed it. Why should some things xx *not* form a set? The semantics of plural quantification ensures that it is determinate which things are among xx. And a set is completely characterized by specifying its elements. We can thus give a complete and precise characterization of the set that xx would form if they did form a set. What more could be needed for such a set to exist?[5] Unfortunately, as we will see shortly, COLLAPSE appears to lead straight to paradox.

In this paper I investigate the arguments that can be offered for and against COLLAPSE. I begin by considering what is widely thought to be a knock-down argument against COLLAPSE, namely the charge that it leads to paradox (Section 2). I argue that this charge is much too quick. The paradox shows that one of our assumptions must be mistaken. But it does not immediately establish that the culprit is COLLAPSE; there are other possible culprits as well. The question of the viability of COLLAPSE thus remains open. Next I develop an argument in favor of COLLAPSE (Section 3). Then I consider the main alternative to COLLAPSE, namely the view that some things form a set just in case they are not "too many." I argue that this view is committed to an arbitrary boundary between pluralities that do and do not form sets (Section 4). I end by developing a new response to the threat of paradox, based on the view that the quantifiers used in set theory have an implicit modal aspect (Sections 5 and 6).

2 Whence the contradiction?

COLLAPSE appears to open the door to a version of Russell's paradox. We see this as follows. The *plural comprehension scheme* of ordinary plural logic says that for any formula $\phi(u)$ there

[5]It is thus not surprising that COLLAPSE and related views have been are defended by a number of logicians and philosophers. See for instance Michael Dummett, *Frege: Philosophy of Language* (Cambridge: Harvard, 1981), ch. 15; Kit Fine, "Our Knowledge of Mathematical Objects," in Tamar S. Gendler and John Hawthorne, eds., *Oxford Studies in Epistemology* (New York: Oxford, 2005), pp. 89-109; Kurt Gödel, "The Present Situation in the Foundations of Mathematics," lecture given in 1933, repr. in Solomon Feferman *et al*, eds., *Collected Works* (New York: Oxford, 1995), vol. III, p. 47; Geoffrey Hellman, *Mathematics without Numbers* (New York: Oxford, 1989), ch. 2; Parsons, "Sets and Classes," *Noûs*, VIII (1977): 1-12, "What Is the Iterative Conception," and *Mathematics in Philosophy*, ch. 11; Hilary Putnam, "Mathematics without Foundation," this Journal, LXIV: 5-22; Stephen Yablo, "Circularity and Paradox," in Thomas Bolander *et al*, eds., *Self-Reference* (Stanford: CSLI, 2006), Section 5; and Ernest Zermelo, "Über Grenzzahlen und Mengenbereiche", *Fundamenta Mathematicae*, XVI (1930): 29-47, transl. in William Ewald, *From Kant to Hilbert* (New York: Oxford, 1996), vol. 2.

are some things xx which are all and only the things that satisfy this formula:[6]

(P-Comp) $$\exists xx \forall u[u \prec xx \leftrightarrow \phi(u)]$$

An instance of this scheme allows us to consider the objects rr that include all and only the sets that are not elements of themselves:

(1) $$\forall u(u \prec rr \leftrightarrow u \notin u)$$

By COLLAPSE, rr form a "Russell set":

(2) $$\exists r \forall u(u \in r \leftrightarrow u \prec rr)$$

From (1) and (2) we get the following characterization of the Russell set r:

(3) $$\forall u(u \in r \leftrightarrow u \notin u).$$

By instantiating the quantifier $\forall u$ with respect to r, we then get:

(4) $$r \in r \leftrightarrow r \notin r$$

Given any reasonable propositional logic, this yields a contradiction. One of the assumptions of the argument will therefore have to go.

This argument is widely believed to be a refutation of COLLAPSE. In particular, the argument is believed to show that the "Russell plurality" rr fails to form a set.

In fact, defenders of this view typically make the stronger claim that the overwhelming majority of pluralities do not form sets. This claim is based on a generalization of Cantor's theorem. The original theorem says that, for any set x, there are more subsets of x than there are elements of x. The generalization says that, for any plurality xx, there are more subpluralities of xx than there are objects among xx. This generalization entails that the overwhelming majority of pluralities do not form sets. To see this, apply the generalization to the plurality of absolutely all objects. It follows that there are more pluralities than there

[6]The variable 'u' is assumed to be free in ϕ. More generally, any variables enclosed in parentheses after a designation of a formula will be assumed to be free in the formula, which may contain other free variables as well.

are objects and thus *a fortiori* more pluralities than there are sets.

These two arguments are often thought to constitute a knock-down argument against Collapse. But this is much too quick! Admittedly, every step of each of the two arguments is *prima facie* plausible. But Collapse too enjoys *prima facie* plausibility. Since a set is completely characterized by its elements, any plurality xx seems to provide a complete and precise characterization of a set, namely the set whose elements are precisely xx. What more could be needed for such a set to exist? We are thus confronted with a paradox, in the sense that we have made a number of assumptions, all of which seem perfectly plausible, but which jointly lead to contradiction. This calls for careful a scrutiny of the assumptions, not a rushed rejection of one of them. Any attempt to block the paradox by giving up one of the assumptions must be accompanied by an explanation of why *this particular* assumption is the one that should be given up. Is the apparent plausibility of the assumption deceptive? Or is the assumption less compelling than the others? In the absence of such an explanation, the suggestion that we give up one of the assumptions will be no more than an *ad hoc* trick to avoid contradiction. In order to find the true culprit, all suspect assumptions must be identified and given a fair trial.

What other assumptions involved in the two arguments may be suspect? Consider first the argument based on Russell's paradox. In addition to Collapse, at least two other assumptions have been found suspect. First there is the premise (1), which allows us to form the Russell plurality consisting of all and only the non-self-membered sets. Why should it be possible to collect together, once and for all, all and only the things that satisfy the condition of being non-self-membered?[7] Perhaps this question has a good answer. But until that has been established, it will be premature to blame the contradiction on Collapse.

Another possible culprit is the assumption that it is possible to quantify over absolutely everything.[8] If this cannot be assumed, then the range of the quantifiers may have changed in the course of the argument. In particular, it can be argued that the quantifier $\exists r$ in (2) has a greater range than the quantifier $\forall u$ in (1). If so, the final step from (3) to (4) will not be valid, as it will involve the instantiation of a universal quantifier with respect to an object that cannot be assumed to be in its range. So again, until this alternative culprit has been acquitted, it will be premature to blame the contradiction on Collapse.

[7]See for instance Yablo, *op. cit.*

[8]This assumption is challenged in Dummett, *op. cit.*, ch. 16; Parsons, "Sets and Classes" and "What Is the Iterative Conception"; and Michael Glanzberg, "Quantification and Realism," *Philosophy and Phenomenological Report*, LXIX (2004): 541-72.

Consider next the argument based on the generalization of Cantor's theorem. The proof of this generalization relies crucially on an instance of the plural comprehension scheme. But this scheme is the first of the two possible alternative culprits identified above. If this assumption is called into question, the desired proof can no longer be assumed to go through. What about the second suspect assumption, that we can quantify over absolutely everything? The generalization of Cantor's theorem does not rely on this assumption. However, if the quantifiers involved in the generalization cannot be assumed to range over absolutely everything, there will no longer be any conflict with Collapse. For it may then be the case that the range of the quantifiers expands whenever Collapse is affirmed.

I conclude that despite the two arguments against Collapse, the question of its status remains wide open. These are paradoxical arguments that need to be addressed by examining the strengths and weaknesses of all of the assumptions that conspired to generate the contradiction. However, since the majority view is almost certainly that the weakest link is Collapse, much of my discussion will focus on that assumption.

3 In favor of Collapse

Let us now take a closer look at the argument in favor of Collapse that was adumbrated above. The argument observed that pluralities and sets fit naturally together. The semantics of plural logic ensures that a plurality consists of a determinate range of objects. But a set is completely characterized by its elements. A plurality thus provides the resources for a complete and precise characterization of a set. So what could prevent us from collecting the given plurality into a set? It seems we should be able to use any given plurality xx to define a set y as follows:

(5) $$\forall u(u \in y \leftrightarrow u \prec xx)$$

To spell out this argument we must examine more carefully the idea that a set is completely characterized by its elements. The concept of a set is roughly the concept of a collection "constituted by" its elements. One aspect of this idea is that it is part of the nature of a set what elements it has. In particular, two sets cannot be identical unless they have the same elements. Another aspect is the converse, namely that the nature of a set is exhausted by what elements it has. Once you specify the elements of a set, you have specified everything that is essential to it. Every other property of the set flows from its having precisely these elements. In particular, if two sets have the same elements, then they are identical. A third aspect is that the elements of a set are "prior to" the set itself. If a set is a collection constituted by its elements, then these elements will have to be available before the set itself can be formed.

I claim that definition (5) is compatible with all these aspects of the concept of a set. First, the definition specifies the elements of the set y as determinately as the plurality xx itself is specified. (And if the plurality xx isn't determinately specified, it cannot serve as a counterexample to COLLAPSE.) Next, the definition does not ascribe to y any properties beyond having precisely xx as elements. Finally, the definition is compatible with the idea that the set y is formed only "after" its elements xx. In particular, there is no pressure to say that y is one of xx, which would entail the problematic claim that y is an element of itself.

To appreciate the last point, it is instructive to contrast the definition of a set whose elements are given by a plurality xx with an attempted definition of a set whose elements are given by a concept F:

$$\forall u(u \in z \leftrightarrow Fu) \tag{6}$$

For most concepts F, definition (6) will be incompatible with the concept of a set. For instance, let F be the concept of being self-identical. If this concept defined a set z, then z would fall under the concept. Thus by (6), z would have to be an element of itself. But this would conflict with an aspect of the concept of a set, namely the priority of the elements of a set to the set itself. Consequently there can be no such set z.

What tends to make definitions of the form (6) problematic is what we may call *the intensional nature* of concepts. The identity of a concept is tied to its condition of application. For instance, the identity of the concept of being a philosopher is tied to its applying to all and only philosophers. It is therefore essential to a concept that it applies to all and only the objects that satisfy the associated condition of application. This is relevant to the question of whether every concept F defines a set in accordance with definition (6). Were there to be a set z of all Fs, then the intensional nature of the concept F may require F to apply to z. For instance, we saw that there is such a requirement when F is the concept of being self-identical. But if F is to apply to z, then z must be an element of itself, which would conflict with the concept of a set.

Where the intensional nature of concepts often prevents concepts from defining sets, what we may call *the extensional nature* of pluralities shields them from the corresponding danger. The identity of a plurality is tied to the individual objects that it includes. So when a plurality is given, it is determined once and for all which individual objects it includes. There are no possible worlds or conceivable circumstances in which a plurality includes other objects than those that it in fact includes; this is part of what it is to be the plurality in question and not some other plurality. This ensures that, if a plurality xx were to form a set y in accordance with definition (5), there will be no pressure to say that y is one of xx. Consequently there is no danger of committing oneself to the problematic claim that y is an element of itself.

4 A viable alternative to Collapse?

Is there any viable alternative to COLLAPSE? If it is not the case that some things always form a set, what *is* required for some things to do so? A clear and well motivated answer to this question would greatly help opponents of COLLAPSE by providing indirect support for their view that the best response to the paradox is to give up COLLAPSE.

The answer that almost all opponents of COLLAPSE give has to do with considerations of size or cardinality.[9] The idea is that set formation is constrained by some principle of limitation of size. Sometimes things are simply too many to form a set. This view has many attractive features. For one thing, the view is principled and relies only on well understood mathematical concepts such as cardinality. For another, the view appears to be part of the conception of sets underlying standard ZFC set theory. To see this, consider the axiom scheme of Replacement, which says that the image of a set under a functional relation is itself a set. Our confidence in this axiom scheme appears to be based on the fact that the image of a collection under such a relation can be no larger than the collection itself.

In fact, it is hard to see how an explanation of why some pluralities fail to form sets can *avoid* appealing to some principle of limitation of size. For as I will now show, two widely held assumptions entail that the dividing line between pluralities that form sets and those that do not has to be a matter of their size or cardinality. More precisely, the two assumptions entail that every plurality below some threshold cardinality forms a set, whereas every plurality at or above this threshold cardinality fails to form a set.

The first assumption is a version of Replacement. In ordinary first-order set theory,

[9] See for instance John P. Burgess, "*E Pluribus Unum*: Plural Logic and Set Theory," *Philosophia Mathematica*, XII (2004): 193-221 and David Lewis, *Parts of Classes* (Oxford: Blackwell, 1991).

Replacement is expressed as an axiom scheme, with one axiom for each formula that expresses a functional relation. But once plural quantifiers are available, Replacement is more naturally expressed as a single axiom. Say that xx are *at least as many as* yy just in case there is a function (represented as a plurality of ordered pairs) from xx onto yy. Then the axiomatic version of Replacement says that, when xx form a set and xx are at least as many as yy, then yy too form a set. Once plural logic is accepted, the axiomatic form of Replacement is just as plausible as the schematic form. Any reason to accept the schematic version appears also to be a reason to accept the axiomatic version.[10]

The second assumption is that the cardinalities of any two pluralities are comparable. Say that xx are *fewer than* yy just in case yy are at least as many as xx but not *vice versa*. Then the principle of *Cardinal Comparability* says that, given any things xx and yy, either xx are fewer than yy or xx are at least as many as yy. I explain in an appendix how this principle follows from other principles that are widely accepted by writers on plural logic.

When Cardinal Comparability is combined with the Replacement axiom, we get a very informative characterization of the dividing line between pluralities that form sets and those that do not.

Fact 1 Assume plural Replacement and Cardinal Comparability. Then xx form a set if and only if xx are fewer than the ordinals.

(A proof is outlined in the appendix.) It is thus no accident that some version of the principle of limitation of size is the most popular alternative to COLLAPSE. For given the two widely held assumptions, it follows logically that the dividing line between pluralities that do and do not form sets must coincide with the line between the pluralities that are and are not below some threshold cardinality. This makes it reasonable to focus our investigation of alternatives to COLLAPSE on the principle of limitation of size.[11]

Can some version of this principle provide a viable alternative to COLLAPSE? The main challenge will be to motivate and defend the threshold cardinality beginning at which pluralities are too large to form sets. Why should this particular cardinality mark the threshold? Why not some other cardinality?

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[10] Use of an axiomatic form of Replacement has been standard in second-order set theory ever since Zermelo, *op. cit.*

[11] The argument of this paragraph assumes that there is a plurality of absolutely all ordinals. This is an assumption to which opponents of COLLAPSE are firmly committed. My own response to the argument will be to reject this assumption in favor of the view that all pluralities are "set-sized". So I have no need (or desire) to challenge Replacement and Cardinal Comparability.

A preliminary question is how to understand the threshold cardinality. Let *oo* be the plurality of all ordinals. Perhaps the cardinality of *oo* admits of an explicit characterization in plural logic. An explicit characterization can for instance be given if the cardinality of *oo* is the first strongly inaccessible.[12] However, we are under no obligation to provide an explicit characterization of the threshold cardinality. An alternative is to use *oo* as a "measuring stick" and understand the threshold cardinality as the cardinality instantiated by *oo*. This parallels the way in which we can fix a measure of length by reference to the length instantiated by a designated object. Here we fix a measure of cardinality by reference to a designated plurality.

Our question is thus why the cardinality instantiated by *oo* should mark the threshold beginning at which pluralities are too large to form sets. One answer is that it is somehow "written into" the concept of set that every set must have fewer elements than this threshold cardinality. But how should a reference to this particular cardinality have managed to be written into the concept of a set? This is certainly not anything that *we* have brought about, say by defining 'set' as a collection of size smaller than this cardinality. Another possibility is that the restriction is built into some objective and pre-existing concept of a set, which we have merely discovered and acquired a more or less adequate grasp of. But then one wonders how this particular cardinality could have come to play this important role in the objective concept of set. True, some cardinals serve as natural milestones in the hierarchy of sets. For instance, the inaccessible cardinals provide natural milestones. But given any inaccessible cardinal, there is no logical or mathematical obstacle to going on to define sets of even larger cardinalities.[13] This is also borne out by the history of set theory. Wherever it has been possible to go on to define larger sets, set theorists have in fact done so. So it remains arbitrary that there should be no sets of this cardinality or some even larger one.[14]

It may be responded that the proposed threshold at and above which pluralities are too large to form sets is a very special cardinality because it derives from a very special plurality, namely the plurality *oo* of all ordinals. Since this plurality is special and non-arbitrary, so is its cardinality.

However, it is not clear that the plurality *oo* is special and non-arbitrary. No doubt there is something special and non-arbitrary about the concept of an ordinal. But why should

[12]See Stewart Shapiro, *Foundations without Foundationalism: A Case for Second-Order Logic* (New York: Oxford, 2000), p. 106.

[13]This view is supported by a vast number of logicians and philosophers; see footnote 5 for a sample.

[14]Indeed, in Morse-Kelly set theory we add 'classes' of this cardinality, which are just like sets except for the artificial prohibition against their being elements of sets or classes.

this non-arbitrariness be inherited by the plurality *oo* of objects falling under the concept? Compare the concept of a human being, which we may concede is special and non-arbitrary. This non-arbitrariness is not inherited by the plurality *hh* of all human beings. For there could easily have been other human beings than those there actually are, in which case humankind would have comprised some plurality other than *hh*. Since the non-arbitrariness of a concept need not be inherited by the plurality of objects falling under the concept, we have not yet been given any reason to believe that the plurality *oo* of all ordinals is non-arbitrary.

To probe further, consider the question why there are not more ordinals than *oo*. For instance, why cannot the plurality *oo* form a set, which would then be an additional ordinal, larger than any member of *oo*? According to the view under discussion, the explanation is that *oo* are too many to form a set, where being too many is defined as being as many as *oo*. So the proposed explanation moves in a tiny circle. The threshold cardinality is what it is because of the cardinality of the plurality of all ordinals; but the cardinality of this plurality is what it is because of the threshold. I conclude that the response fails to make any substantial progress, and that the proposed threshold remains arbitrary.[15]

5 How best to restore consistency

We have seen that Collapse, in conjunction with some other assumptions, leads to a contradiction. The standard response has been to reject Collapse and keep the other assumptions. But this response is unappealing in light of the above discussion, which shows there to be good arguments in favor of Collapse and no good alternatives to it. I will now examine whether there is any better way to restore consistency.

One alternative to the rejection of Collapse is to weaken the plural comprehension scheme:

(P-Comp) $$\exists xx \forall u[u \prec xx \leftrightarrow \phi(u)]$$

This response thus denies that for any formula $\phi(u)$ there are some things that are all and only the ϕ's. As explained in Section 2, this response is easily seen to restore consistency.

[15] There are other reasons too to worry that the ascription to the universe of any definite cardinality will be problematically arbitrary. For instance, Gabriel Uzquiano, "Unrestricted Unrestricted Quantification," in Agustín Rayo and Gabriel Uzquiano, eds., *Absolute Generality* (New York: Oxford, 2006), pp. 305-332, describes several pairs of *prima facie* plausible theories where each pair of theories require the universe to satisfy incompatible cardinality requirements.

The hard part is to defend the superiority of this response over the standard one of rejecting Collapse. Many philosophers will no doubt think that any weakening of (P-Comp) is a complete non-starter. How could a determinate condition possibly fail to define a plurality?

This question receives an interesting answer from Stephen Yablo, who is one of the few theorists to have explicitly defended this alternative response to the paradox.

> The condition $\phi(u)$ that (I say) fails to define a plurality can be a perfectly determinate one; for any object x, it is a determinate question whether x satisfies $\phi(u)$ or not. How then can it fail to be a determinate matter what are *all* the things that satisfy $\phi(u)$? I see only one answer to this. Determinacy of the ϕ's follows from
>
> (i) determinacy of $\phi(u)$ in connection with particular candidates,
>
> (ii) determinacy of the pool of candidates.
>
> If the difficulty is not with (i), it must be with (ii).[16]

How can there fail to be a determinate pool of candidates? According to Yablo, the answer has to do with the iterative conception of sets. The universe of sets is built up in stages. At each new stage we introduce all the sets that can be formed from the objects available at the preceding stages. But there is no stage at which this process of set formation is complete. At any one stage it is possible to go on and form new and even larger sets.

This very suggestive answer needs to be spelled out. I will now propose a way of doing so, based on the idea that the hierarchy of sets is a potential one, not a completed or actual one. The existence of a set is potential relative to its elements: if the elements exist, their set may be formed.[17] To make explicit this potential character of the hierarchy of sets, we introduce two modal operators $\Diamond$ and $\Box$. Officially, these operators are new primitives governed only by a modal logic. But for heuristic purposes it will be useful to interpret $\Diamond\phi$ as "it is possible to go on to form sets so as to make it the case that ϕ", and $\Box\phi$ as "no matter what sets we go on to form it will remain the case that ϕ."[18]

I claim that the quantifiers used in set theory should be seen as having an implicit modal character. When a set theorist writes $\forall$ and $\exists$, she should typically by analyzed as meaning

[16]Yablo, *op. cit.*, pp. 151-2; some notation and terminology has been adapted to the conventions of this article.

[17]See Parsons, "What Is the Iterative Conception," for a closely related view.

[18]A similar view on the set theoretic quantifiers is suggested in Putnam, *op. cit.*, p. 21 and developed in Hellman, *op. cit.*, pp. 73-9. Parsons, *Mathematics in Philosophy*, pp. 321-2 suggests an alternative translation based on the standard translation of intuitionistic logic into the modal logic S4.

$\Box\forall$ and $\Diamond\exists$ respectively. So when a set theorist says that a formula holds for "all sets," she should typically be understood as claiming that no matter how far the hierarchy of sets is continued, the formula will hold of all the sets formed by then. And when a set theorist says that a formula holds for "some set," she should typically be understood as claiming that it is possible to continue the hierarchy of sets such that there is some set of which the formula holds. We can make this implicit modal character of the quantifiers explicit by means of a translation from the language of non-modal set theory into the language of modal set theory. Given a non-modal formula ϕ, we let its translation $\phi^{\Diamond}$ be the result of replacing each occurrence of $\forall$ and $\exists$ in ϕ with respectively $\Box\forall$ and $\Diamond\exists$.

What is the status of this proposed analysis of the set theoretic quantifiers? The analysis is not meant as a contribution to linguistics or the philosophy of language. My only claim concerning linguistics and the philosophy of language is that the analysis is compatible with the empirical data. This compatibility is based on the fact that in modal set theory the composite expressions $\Box\forall$ and $\Diamond\exists$ behave logically just like quantifiers, as will be explained in an appendix. My reason for proposing the analysis is that it is philosophically and mathematically illuminating. It allows us to take seriously the potential character of the set theoretic hierarchy, while still recognizing that set theorists routinely make assertions about "all sets" regardless of the stage at which they are formed.

I can now state my preferred response to the paradox. In any discussion that involves set theory, the quantifiers $\forall$ and $\exists$ can be understood in two different ways: either as having an implicit modal character or not. On the former understanding, the quantifiers function as shorthand for respectively $\Box\forall$ and $\Diamond\exists$. On the latter understanding, the quantifiers are rightly taken at face value. I claim that the paradox results from an equivocation between these two understandings of the quantifiers. We have seen that a contradiction ensues when both Collapse and unrestricted plural comprehension (P-Comp) are affirmed. But on each understanding of the quantifiers, one of these assumptions fails. When the quantifiers are understood as having an implicit modal character, Collapse is true and (P-Comp) is false. And when the quantifiers are taken at face value, Collapse is false and (P-Comp) is true.

These claims need to be defended. Assume first that the quantifiers are understood as not having an implicit modal character and are thus taken at face value. Then I claim that Collapse is false. For given some objects xx, the existence of a set with precisely xx as elements is only potential. This means that we *may* go on to form such as set. But it does not mean that such a set has *actually* been formed, which is what collapse asserts. Next I claim that on this way of understanding the quantifiers, (P-Comp) is true. For at each stage of the process of forming sets there is a determinate range of sets. (In Yablo's terminology, there is "a determinate pool of candidates.") This ensures that a condition $\phi(u)$ whose meaning is determinate succeeds in defining a determinate plurality of all and only the ϕ's.

Assume next that the quantifiers are understood as having an implicit modal character, as I have argued is often the case in set theory. This modal character is made explicit in the modal translations of the two controversial assumptions. The modal translation of COLLAPSE is:

(COLLAPSE$^\Diamond$) $$\Box\forall xx\Diamond\exists y\textsc{Form}(xx, y)$$

So on this way of understanding the quantifiers, COLLAPSE is elliptical for COLLAPSE$^\Diamond$, which says that given any objects xx, *it is possible* to form a set whose elements are precisely these objects. This claim is true on the modal analysis of the iterative conception of sets that I outlined above. For on this conception, any sets that have been formed may be used to form a new set. Indeed, as Yablo nicely puts it, COLLAPSE$^\Diamond$ is on this conception "the principal engine of set production" (Yablo, *op. cit.*, p. 151).

The modal translation of the unrestricted plural comprehension scheme is:

(P-Comp$^\Diamond$) $$\Diamond\exists xx\Box\forall u[u \prec xx \leftrightarrow \phi(u)]$$

So this scheme translates as the claim that, given any formula $\phi(u)$ it is possible for there to be some objects xx such that *no matter what sets we go on to form*, xx are all and only the ϕ's. It is not hard to see that this claim is problematic. Recall from Section 3 that pluralities have an extensional nature and that it is thus essential to a plurality which objects it includes. So in any world in which a particular plurality exists it must include precisely the same objects. But (P-Comp$^\Diamond$) makes claims about pluralities that conflict with their extensional nature. Consider for example the following instance of (P-Comp$^\Diamond$)

(7) $$\Diamond\exists xx\Box\forall u[u \prec xx \leftrightarrow u = u]$$

which says that it is possible for there to be some objects xx such that no matter what sets we go on to form, xx are all and only the self-identical objects. But this is impossible. As we go on to form more sets, there will be more and more self-identical objects. But by the extensional nature of pluralities, the objects xx will consist of precisely the same objects even as we go on to form more sets. Consequently (7) is false.

Many other instances of (P-Comp$^\Diamond$) are false as well. Another example is the modal translation of assumption (1) from the derivation of Russell's paradox in Section 2:

$$(1^\Diamond) \qquad \Diamond\exists xx\Box\forall u[u \prec xx \leftrightarrow u \notin u]$$

For as we go on to form more sets, there will be more and more non-self-membered objects. The good instances of (P-Comp$^\Diamond$) are those where there is some stage w in the process of forming sets at which all possible instances of the formula $\phi(u)$ have been formed.[19] Then $\phi(u)$ will remain true of precisely the same objects no matter what sets we go on to form. In fact, since these objects have all been formed at w, there will be a stage w' later than w at which they form a set. This means that the good instances of (P-Comp$^\Diamond$) are precisely those where the comprehension formula $\phi(u)$ defines a set. We thus have an independent motivation for a weakening of (P-Comp$^\Diamond$) which is compatible with Collapse$^\Diamond$.

6 Some further questions

I now address some further questions raised by the proposal just outlined.

One question is how to understand the modal notions that are invoked. These notions cannot be understood in terms of ordinary metaphysical modality. For the existence of pure sets is generally taken to be metaphysically necessary, whereas the modality invoked above allows the existence of sets to be contingent. This modality must therefore be more fine-grained than metaphysical modality.

Strictly speaking, all we need to assume about the above notion of modality is that it is suited to explicating the iterative conception of set. The modality must thus be one on which the existence of a set is potential relative to the existence of its elements (in the sense that, when some things exist, it is possible for there to exist a set with precisely these things as elements). All other details are optional. Different writers can and do spell out the ideas in somewhat different ways.[20]

Let me nevertheless mention my preferred way of spelling out the details. I understand the

[19] See Hellman, *op. cit.*, pp. 72-3 for a more severe restriction on an analogous modal second-order comprehension scheme.

[20] See for instance Fine, *op. cit.*; Hellman, *op. cit.*; and Parsons, *Mathematics in Philosophy*, ch. 11.

above modalities in terms of a process of individuating mathematical objects. To individuate a mathematical object is to provide it with clear and determinate identity conditions. This is done in a stepwise manner, where at any stage we can make use of any objects already individuated in our attempts to individuate further objects. In particular, at any stage we can consider a plurality of objects already individuated and use this to individuate the set with precisely these objects as elements. A situation is deemed to be possible relative to one of these stages just in case the situation can be obtained by some legitimate continuation of the process of individuation.

A related question is which sets are actual with respect to the modality invoked above. One answer is that precisely those sets are actual that have been formed at our current stage of the process of forming sets. But this is uninformative. For what *is* our current stage of this process? The most plausible response to this follow-up question is, I think, that set theorists do not generally regard themselves as located at some particular stage of the process of forming sets but rather take an external view on the entire process. It would therefore be wrong to assign to ourselves any particular stage of the process.[21]

A third question is whether my account can be used to interpret the influential but enigmatic suggestions made in Zermelo, *op. cit.*. Here Zermelo proposes "the general hypothesis that *every categorically determined domain can also be interpreted as a set in some way*" (p. 1232, his emphasis). Since the domains in question are described using a second-order language, they correspond to pluralities in our present setting. Translated into our setting, Zermelo's proposal is thus that every plurality that is determined in a certain way can be interpreted as a set. In particular, any plurality that provides a model of second-order ZF set theory can be interpreted as a set in some larger model of this set theory. Zermelo thus seems to be committed to a claim very much like COLLAPSE. How can he then avoid contradiction?

My account of the relation between pluralities and sets points to an interesting reading of Zermelo. Perhaps what Zermelo proposed was not COLLAPSE but its modalized counterpart COLLAPSE$^{\Diamond}$.[22] Some evidence for this reading is provided by the fact that Zermelo uses modal vocabulary to describe the transition from a "categorically determined domain" to the corresponding set. However, since this is not the place for a systematic discussion of Zermelo's

[21] An interesting alternative response is the following. As science progresses, we formulate set theories that characterize larger and larger initial segments of the universe of sets. At any one time, precisely those sets are actual whose existence follows from our strongest well-established set theory.

[22] More precisely, although Zermelo's object theory is most naturally interpreted as using ordinary non-modal quantifiers, perhaps his meta-theory is best interpreted as using implicitly modalized quantifiers.

work, I restrict myself to the claim that my proposed reading provides an interesting way of making sense of Zermelo and that it enjoys some *prima facie* textual evidence.

The next question is more critical. If I am right that every plurality potentially defines a set, then there are fewer pluralities than one would naively have thought. One would have thought that there could be pluralities that are not exhausted by any given level of the set theoretic hierarchy. But my account leaves no room for such pluralities, requiring instead that every plurality be exhausted by some possible level. It will therefore be objected that my restrictions on the existence of pluralities undercuts some of the most useful applications of plural logic. For instance, plural logic has been used to develop the semantics of first-order set theory where the first-order quantifiers are allowed to keep their intended range over absolutely all sets. This application relies crucially on our ability to represent the domain by means of a plurality consisting of absolutely all sets. By disallowing such pluralities, I disallow some useful applications of plural logic.[23]

I grant that such applications are disallowed but deny that this represents any loss. For one thing, you cannot lose something you never had. And this was never a permissible application of plural logic. The extensional nature of pluralities makes them unsuited to represent the potential hierarchy of sets. A plurality consists of a fixed range of objects. But the set theoretic hierarchy is inherently potential in its nature and thus resists being summed up by a fixed range of objects. For another thing, the job in question is better and more naturally done by means of entities such as Fregean concepts. The intensional nature of concepts makes them well suited to represent the domain of set theory or the semantic value of the membership predicate.

A final question is whether on my account there remains any important theoretical role for plural logic. Given that many philosophers and logicians are already interested in both plural logic and set theory, the question naturally arises how best to combine the two. My account provides an answer to this question. But suppose you do not have any antecedent interest in plural logic. Does my account give you any reason to become interested in plural logic? I believe the answer is 'yes'. By introducing plural logic we can make explicit the relation between a set and the objects from which the set is formed, which is useful in discussions of the foundations of set theory. In particular, the availability of plural logic allows us to provide a very elegant motivation of ordinary Zermelo-Fraenkel set theory.[24]

[23] I have in mind the project initiated by Boolos, "Nominalist Platonism," *Philosophical Review*, XCIV (1985): 327-344.

[24] See my "The Potential Hierarchy of Sets" (unpublished manuscript), which is inspired by Parsons, *Math-*

7 Conclusion

The nature of our paradox was this. On the one hand COLLAPSE is very difficult to avoid. On the other hand COLLAPSE leads to contradiction in the presence of some other widely held assumptions. I have proposed a solution to this paradox based on the idea that the set theoretic hierarchy is potential and that the quantifiers used in set theory therefore often have an implicit modal character. In connection with set theory there are thus two different ways of understanding the quantifiers. The paradox results from equivocating between these two understandings. On each uniform understanding of the quantifiers one of the assumptions responsible for the contradiction has been shown to fail.

This solution has three desirable features. Firstly, it allows us to hold on to COLLAPSE provided its quantifiers are understood in the usual set theoretic way as having an implicit modal character. Secondly, it allows us to hold on to the unrestricted plural comprehension scheme (P-Comp) provided its quantifiers are understood in the way that is most usual outside of set theory, namely as not having an implicit modal character. Thirdly, this solution provides an independently motivated account of why the plural comprehension scheme must be restricted when (such as in the context of set theory) its quantifiers are understood as having an implicit modal character.

Appendix

The principle of Cardinal Comparability

Global Well-Ordering (GWO) says that there are some ordered pairs that form a well-ordering of the universe. Standard techniques show that GWO implies Cardinal Comparability (CC): Given two pluralities xx and yy, we can define some ordered pairs which correlate objects from xx with objects from yy in the order induced by the global well-ordering and which give an injection of either xx into yy or *vice versa*.[25]

Writers on plural logic and set theory tend to be implicitly committed to GWO. This may happen because of a commitment to the ordinary set theoretic Axiom of Choice and the so-called *reflection principle*, which says that any property of the set-theoretic universe is reflected in some set-sized sub-universe. (See Burgess, "*E Pluribus Unum.*") For if GWO

ematics in Philosophy, ch. 11.

[25] I do not know whether CC implies GWO. If not, there may be routes to CC which are even less controversial than those described below.

failed, this property would then "reflect down" and yield a counterexample to the claim that any set can be well-ordered, which follows from the Axiom of Choice. Since both the Axiom of Choice and the reflection principle are widely accepted, so is (if only implicitly) GWO.

Another way in which people tend to be implicitly committed to GWO is via the assumption that the *urelemente* form a set and a principle known as *Global Choice* (GC), which in our current setting says that there is a plurality that codes a function that maps each non-empty set x to an element of x. Given the former assumption and ZF set theory, it follows that the universe is a union of ranks. Given GC, we can choose a well-ordering of each rank. Then the universe is well-ordered by the lexicographic order of rank and the chosen well-ordering within each rank.

Sketch of proof of Fact 1

For the left to right direction, assume that xx form a set but that it is not the case that xx are fewer than the ordinals. Then xx are at least as many as the ordinals (by Cardinal Comparability). But then the ordinals form a set (by Replacement). By the Burali-Forti paradox, this leads to contradiction. For the other direction, assume that xx are fewer than the ordinals. Then there is a surjection of the ordinals onto xx, which induces a well-ordering on xx. This well-ordering is either isomorphic to the elements of some (set) ordinal (with their natural ordering) or to the ordinals themselves (with their natural ordering). Since the latter would contradict the assumption that xx are fewer than the ordinals, the former must be the case. Then Replacement yields that xx form a set.

Logical properties of the modalized quantifiers

The strings $\Box\forall$ and $\Diamond\exists$ are strictly speaking composite expressions, not quantifiers. I will now show that in the context of an appropriate modal set theory, these composite expressions nevertheless behave logically much like ordinary quantifiers. This will justify my convention of referring to these composite expressions as *modalized quantifiers*.

Let L be the plural logic based on classical sentential logic, the standard axioms of identity, and the standard introduction and elimination rules for the (singular and plural) quantifiers, but without any plural comprehension axioms.[26] It is important that the elimination rules for the plural quantifiers be formulated so as to allow only variables as instances. For example,

[26] For instance, let P be the logic PFO of Linnebo, "Plural Quantification," minus the plural comprehension scheme.

from $\forall xx(a \prec xx)$ we can directly infer that $a \prec yy$ but not that a is one of the ordinals. The latter inference must proceed via the comprehension axiom $\exists yy \forall x(x \prec yy \leftrightarrow x$ is an ordinal), which makes explicit the assumption that 'x is an ordinal' succeeds in defining a plurality.

Next I describe the appropriate modal logic. I am interested in a system of possible worlds whose domains consist of the sets individuated thus far. Since the individuation of a set consists in specifying its elements, each world must also represent how the objects in its domain are related by the membership relation $\in$. Next, one world w' is accessible from another w just in case we can get from w to w' by individuating more sets. This means that the accessibility relation must be a partial order (that is, reflexive, anti-symmetric, and transitive); I will therefore write it as $\leq$. What if we have a choice what sets to go on and individuate? Assume we are at a world w_0 where we can individuate further sets so as to arrive at either w_1 or w_2. It makes sense to require that the licence to individuate a possible set never goes away as we build up the hierarchy of sets but can always be exercised at a later stage. This corresponds to the requirement that the two worlds w_1 and w_2 can be extended to a common world w_3. This property of a partial order is called *directedness*. The appropriate modal logic for models whose accessibility relation is a directed partial order is a system known as S4.2, which arises from the familiar system S4 by adding the axiom:

(G) $$\Diamond\Box p \rightarrow \Box\Diamond p.$$

In the quantified modal logic that results from adding S4.2 to our plural logic L, we can prove that any identity holds of necessity. But we cannot prove that any non-identity holds of necessity, although this is widely assumed. Moreover, I argued above that all facts about set membership hold of necessity as well. We therefore add the following axioms (which we refer to as *stability axioms*):

$$x \neq y \rightarrow \Box x \neq y$$
$$x \in y \rightarrow \Box x \in y$$
$$x \notin y \rightarrow \Box x \notin y$$

as well as analogous axioms concerning the relation $x \prec yy$ (that is, x is one of yy). Let T be the theory that results from the plural logic L by adding S4.2 and the stability axioms.[27]

[27] In modal set theory one sometimes needs axioms that prevent sets and pluralities from picking up new members in possible worlds with larger domains; see Parsons, *Mathematics in Philosophy*, p. 302 and Linnebo,

Say that a formula is *fully modalized* iff all of its quantifiers are fully modalized.

Lemma 1 Let ϕ be a fully modalized formula. Then T proves the equivalence of $\Diamond\phi$, ϕ, and $\Box\phi$.

Proof sketch. Assume ϕ is fully modalized. Since we are working in an extension of the modal logic T, it suffices to prove $\Diamond\phi \to \Box\phi$, which we do by induction on the complexity of ϕ. If ϕ is atomic, then the stability axioms allow us to prove our target $\Diamond\phi \to \Box\phi$. If ϕ is $\neg\psi$, then our target follows by applying the induction hypothesis to ψ. If ϕ is $\psi_1 \wedge \psi_2$, then $\Diamond\phi$ implies $\Diamond\psi_1 \wedge \Diamond\psi_2$, which by the induction hypothesis implies $\Box\psi_1 \wedge \Box\psi_2$, which in turn implies $\Box\phi$. Assume finally that ϕ is of the form $\Diamond\exists x\, \psi$. Since we work in an extension of S4, we have $\Diamond\phi \to \phi$. So it suffices to prove $\phi \to \Box\phi$. Our induction hypothesis yields that ϕ is equivalent to $\Diamond\exists x\, \Box\psi$. Observe next that the Converse Barcan Formula (which is provable in our modal logic) yields $\forall x\Box\exists y(x = y)$ and thus also $\exists x\Box\psi \to \Box\exists x\, \psi$. The formula $\Diamond\exists x\, \Box\psi$ thus implies $\Diamond\Box\exists x\psi$. By (G) the latter formula implies $\Box\Diamond\exists x\, \psi$, which is just $\Box\phi$, as desired. ⊣

Lemma 1 means that it does not matter at which world a fully modalized formula is evaluated, as the result will always be the same. We may thus omit mention of the world at which the evaluation takes place. This is part of the reason why the implicit modalities that I have postulated in the set theoretic quantifiers do not surface in ordinary set theoretic practice.

The next theorem provides a precise statement of my claim that in the context of our modal set theory the composite expressions $\Box\forall$ and $\Diamond\exists$ behave logically just like ordinary quantifiers.

Theorem 1 Let $\phi_1, \ldots, \phi_n$ and ψ be any formulas of non-modal plural set theory. Then

$$\phi_1, \ldots, \phi_n \vdash_L \psi \quad \text{iff} \quad \phi_1^\Diamond, \ldots, \phi_n^\Diamond \vdash_T \psi^\Diamond.$$

Proof sketch. The proof goes by induction on the proofs. We start with the left-to-right direction. The only hard cases are the introduction and elimination rules for the quantifiers. I will outline the case of the singular universal quantifier; the cases of the other quantifiers are analogous. We begin with the rule UI. Assume we have $\phi_1, \ldots, \phi_n \vdash_L \forall x\, \psi$ and conclude by UI that $\phi_1, \ldots, \phi_n \vdash_L \psi(t)$ for some suitable term t. By the induction hypothesis we have $\phi_1^\Diamond, \ldots, \phi_n^\Diamond \vdash_T \Box\forall x\, \psi^\Diamond$, from which we can obviously get $\phi_1^\Diamond, \ldots, \phi_n^\Diamond \vdash_T \psi^\Diamond(t)$. Next

"The Potential Hierarchy." Such axioms are not needed for the purposes of what follows.

we consider the rule UG. Assume that we have $\phi_1, \ldots, \phi_n \vdash_L \psi(t)$ and conclude by UG that $\phi_1, \ldots, \phi_n \vdash_L \forall x\, \psi$. By the induction hypothesis we have $\phi_1^\Diamond, \ldots, \phi_n^\Diamond \vdash_T \psi^\Diamond(t)$, from which UG gives us $\phi_1^\Diamond, \ldots, \phi_n^\Diamond \vdash_T \forall x\, \psi^\Diamond$. A standard trick available in S4 then gives us $\Box\phi_1^\Diamond, \ldots, \Box\phi_n^\Diamond \vdash_T \Box\forall x\, \psi^\Diamond$. Lemma 1 then gives us $\phi_1^\Diamond, \ldots, \phi_n^\Diamond \vdash_T \Box\forall x\, \psi^\Diamond$, as desired.

For the right-to-left direction it is useful to adopt an axiomatic approach to our modal logic. Consider the operation $\phi \mapsto \phi^-$ of deleting all modal operators. This operation maps every axiom of T to a theorem of L and correlates every inference rule of T with a legitimate inference of L. The right-to-left direction then follows from the observation that $(\phi^\Diamond)^- = \phi$. $\dashv$

Theorem 1 tells us that, if we are interested in logical relations between fully modalized formulas in a modal set theory which includes T, we may delete all the modal operators and proceed by the ordinary non-modal logic encapsulated in L. This is another reason why the implicit modalities that I have postulated in the set theoretic quantifiers do not surface in ordinary set theoretic practice.