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*Plural Quantification Exposed**

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English contains two sorts of object quantifiers. In addition to ordinary *singular* quantifiers, as in the sentence 'There is a Cheerio in the bowl', there are *plural* quantifiers, as in 'There are some Cheerios in the bowl'. In the 1980s, George Boolos showed how these plural quantifiers can be used to interpret monadic second-order logic (henceforth *MSOL*).[1] This is an indisputable technical result. However, as with most technical results, it is not obvious what its philosophical cash value is.

According to Boolos, its cash value is that MSOL *really is logic*. In particular, the result is said to show that MSOL introduces no new ontological commitments: that no sets or classes or Fregean concepts are needed as values of the second-order variables, but that the entities already in the first-order domain suffice. Despite some isolated voices of dissent,[2] this philosophical interpretation of Boolos' technical result has become enormously popular. MSOL is now widely regarded as an important part of the philosopher's logical tool kit, of great value not only in the philosophy of mathematics but also in analytic metaphysics more generally. Most of all, Boolos' interpretation of MSOL has been hailed as a way of making great ontological bargains. According to its supporters, it allows us to pay the ontological price of a mere first-order theory and get the corresponding monadic second-order theory for free.[3]

In this paper I will challenge this wide-spread view of the philosophical cash value of Boolos' technical result. I will reject Boolos' claim that this result shows MSOL to be pure logic. And I will express serious misgivings about his claim that MSOL in this way is shown to be ontologically innocent.

To assess what is achieved by Boolos' result, we need to get clear on the *logical status* of the theory of plural quantification. Only if this theory deserves the honorific "logic" will Boolos be justified in claiming that his interpretation shows MSOL to be pure logic. In Section I, I describe the theories of MSOL and of plural quantification and give an informal proof of Boolos' result. Based on this, I attempt to sharpen the question about the logical status of the theory of plural quantification. In Section II, I address the narrower question whether plural quantification is ontologically innocent. This narrower question has been the focus of most recent discussions of Boolos' proposal. I argue that it is a mistake to restrict one's focus in this way and that the question of ontological innocence can be meaningfully addressed only in close connection with the broader question about the logical status of the theory of plural quantification. In Sections III and IV, I therefore turn my attention to this broader question. I argue that a theory of plural quantification that is strong enough to interpret impredicative MSOL will draw heavily on combinatorial and maybe even set theoretic ideas, and that, for this reason, it fails to qualify as logic. Finally, in Section V, I return to the question of ontological innocence. I argue that the preceding discussion leaves us with no clear sense in which plural quantification is ontologically innocent, but on the contrary, that it suggests an interesting sense in which it isn't.

I

Let's begin with an axiomatization of ordinary first-order logic with identity. For our current purposes, it is convenient to axiomatize this logic as a natural deduction system, taking all tautologies as axioms and the familiar natural deduction rules governing the quantifiers and the identity sign as rules of inference. To obtain the language of MSOL, we add to this language second-order variables X_i (for every natural number i) and second-order quantifiers, which may bind these variables. Furthermore, we extend, in the obvious way, the natural deduction rules for the first-order quantifiers to the second-order quantifiers.

To make all existential import explicit, it is useful to adopt the convention that the introduction and elimination rules for the quantifiers are defined on *free variables only*, not on constants. Then, in order to infer '$\exists x.\varphi(x)$' from '$\varphi(t)$' we need the existence claim '$\exists x(x = t)$'.[4] So at the first-order level, this convention means that for each singular term t we need some axiom, for instance '$\exists x(x = t)$', that makes the existential import of 't' explicit. At the second-order level, this convention means that all (apparent) existential import must be made explicit in the form of so-called *comprehension axioms*. These axioms are generated by the comprehension schema

$$\text{(Comp)} \quad \exists X \forall x (Xx \leftrightarrow \varphi),$$

where φ is a formula in the language of MSOL that contains 'x' and possibly other free variables but contains no occurrence of 'X'. Recall that if φ contains no bound second-order variables, the corresponding comprehension axiom is said to be *predicative*; otherwise, it is *impredicative*. (This distinction will play an important role in Section IV.)

Let the *impredicative theory of MSOL* be the theory in the language of MSOL whose axioms are the tautologies and *all* comprehension axioms, and whose rules of inference are the natural deduction rules for identity and for the first- and second-order quantifiers. Let the *predicative theory of MSOL* be the sub-theory containing only the *predicative* comprehension axioms.

In two important papers, Boolos 1984 and 1985, Boolos presents and defends an ingenious interpretation of MSOL. He reminds us that natural languages, such as English, contain two different sorts of quantification. In addition to ordinary singular quantification, there is plural quantification, as in the sentence 'There are some Cheerios in the bowl'. Although the philosophical tradition has been rather oblivious to plural quantification, Boolos argues that this kind of quantification can be used to interpret the second-order quantifiers of MSOL.

Before I present Boolos' argument, I will regiment the plural quantification of English in a formal language that I will call *PFO* (abbreviating *Plural First-Order*).[5] In addition to the usual vocabulary of first-order logic, such as *singular* first-order variables x_i (for every natural number i), PFO contains *plural* first-order variables xx_i and a two-place logical predicate '$\prec$', the first of whose argument-places takes singular arguments, and the second, plural arguments. '$\exists xx_i$' is to be interpreted as the plural quantifier 'there are some objects$_i$ such that', and '$x_i \prec xx_j$' as 'it$_i$ is one of them$_j$'. Following Boolos, I assume that '$\exists xx_i$' is interpreted so as to allow the possibility that the objects$_i$ be just one.

To get a sense of Boolos' interpretation of MSOL in terms of plural quantification, let's begin with an example. Consider the well-known Geach-Kaplan sentence

(GK) Some critics admire only one another.

This sentence is known to be non-first-orderizable: There is no first-order sentence to which it is logically equivalent.[6] In MSOL, (GK) can be formalized as

(1) $\exists X\, (\exists x.Xx \,\&\, \forall y \forall z\, [(Xy \,\&\, \text{Admires}(y,z)) \rightarrow (y \neq z \,\&\, Xz)])$

(where, for simplicity, all variables are restricted to critics). But this sentence can easily be translated into PFO, namely as

(2) $\exists xx\, \forall y\, \forall z\, [(y \prec xx \,\&\, \text{Admires}(y,z)) \rightarrow (y \neq z \,\&\, z \prec xx)]$

(again restricting all variables to critics).

What Boolos discovered was that *any* sentence of MSOL can be translated into PFO. To verify this, we define the following translation mapping *Tr* from MSOL to PFO:

— $Tr(X_j x_i) = x_i \prec xx_j$
— $Tr(\neg\varphi) = \neg Tr(\varphi)$
— $Tr(\varphi \,\&\, \psi) = Tr(\varphi) \,\&\, Tr(\psi)$
— $Tr(\exists x_i \varphi) = \exists x_i . Tr(\varphi)$
— $Tr(\exists X_j \varphi) = \exists xx_j . Tr(\varphi) \vee Tr(\varphi^*)$,
where φ^* is the result of substituting $x_i \neq x_i$ everywhere for $X_j x_i$.

(The second disjunct of the translation of '$\exists X_j \varphi$' is needed to accommodate the case where 'X_j' is interpreted such that no objects fall under it.)

To show that MSOL is pure logic, we need to show that this translation maps the theorems of MSOL to truths of PFO that can be regarded as logical.[7] In order to investigate this, it is useful first to define a *theory* of PFO. What should this theory look like? Since the tautologies of MSOL are mapped to corresponding tautologies of PFO, we begin by adopting all tautologies of PFO as axioms. The second-order comprehension schema are translated as

$$\exists xx\, \forall x\, (x \prec xx \leftrightarrow \varphi') \vee \forall x (x \neq x \leftrightarrow \varphi')$$

or, equivalently but more perspicuously,

$$\text{(Comp-P)} \quad \exists x \varphi' \rightarrow \exists xx\, \forall x\, (x \prec xx \leftrightarrow \varphi')$$

where φ' is in the language PFO and does not contain 'xx' free. I will refer to (Comp-P) as *the plural comprehension schema*. We distinguish between predicative and impredicative plural comprehension axioms in the same way as we did for the second-order comprehension axioms. Next, since all pluralities are non-empty, we adopt '$\forall xx\ \exists x\ (x \prec x)$' as an axiom. Finally, we introduce the natural deduction rules for identity and for the quantifiers, singular as well as plural. We let the quantifier rules be subject to the convention stated above, namely that they be defined on free variables only.

Summing up and being a bit more precise, let *the impredicative theory of plural quantification* be the theory in the language of PFO whose axioms are all PFO tautologies, the statement '$\forall xx\ \exists x\ (x \prec x)$', and *all* plural comprehension axioms, and whose rules of inference are the natural deduction rules for identity and for the singular and plural quantifiers. Let *the predicative theory of plural quantification* be the sub-theory that contains only the *predicative* plural comprehension axioms.

The impredicative theory of plural quantification is defined so as to guarantee that the axioms of the impredicative theory of MSOL are mapped to corresponding axioms of this theory. The same goes for the predicative theory of plural quantification and predicative MSOL. Furthermore, it can be shown that the work done by the natural deduction rules of MSOL can be imitated by the corresponding natural deduction rules of PFO. This establishes Boolos' technical result: that impredicative MSOL can be interpreted in impredicative PFO, and that predicative MSOL can be interpreted in predicative PFO.

We are now finally ready to inquire about the philosophical value of this technical result. Its value depends on the logical status of the impredicative theory of plural quantification. Large parts of this theory can, without serious difficulties, be regarded as purely logical. This holds of the tautologies, the axiom that says that pluralities are non-empty, and the natural deduction rules. The only serious worry concerns the logical status of the plural comprehension axioms. So this is where the battle will be fought over the interpretation of Boolos' technical result.

Boolos contends, with little or no argument, that all of these axioms are logical truths.[8] Clearly, if this contention is correct, then the impredicative theory of plural quantification will qualify as logic. And then Boolos' interpretation of MSOL in PFO will indeed establish that MSOL is pure logic. But why should we accept this contention? The claim that all plural comprehension axioms are logical truths is a very strong one. In particular, this claim should not be confused with the weaker claim that the plural comprehension axioms are *true*. This weaker claim is indeed rather plausible, and I have no quarrel with it: If there is at least one φ, it seems relatively unproblematic to talk about the φs. But this weaker claim does not suffice to establish the strong claim that the plural comprehension axioms are *logical* truths.

Since no general account of the nature and limits of logic is offered or can be presupposed, it's not entirely clear what it means to assert that the theory of plural quantification belongs to pure logic. I suggest that the three following claims can serve as a partial analysis.

> *Ontological Innocence*. The plural comprehension axioms are not ontologically committed to any entities beyond those already accepted in the ordinary first-order domain.

> *Universal Applicability*. The theory of plural quantification can be applied to any realm of discourse, no matter what objects this discourse is concerned with. (This distinguishes the theory of plural quantification both from set theory and from second-order logic with the usual set-theoretic semantics. For although these latter theories can be applied to discourse about any *set-size* domains, they cannot be applied to discourse about domains that are too large to form sets, such as set theory itself.[9])

> *Cognitive Primacy*. The theory of plural quantification presupposes no extra-logical[10] ideas in order to be understood, but can be understood

directly. Our understanding of it does not consist, even in part, in an understanding of extra-logical ideas, such as ideas from set theory or from other branches of mathematics. (Let * be defined as conjunction if the first baby to be born in the twenty-second century is a girl and disjunction otherwise. Cognitive Primacy disqualifies this operator from being logical.)

All of these three claims have to do with logic being unconditioned or presuppositionless: Ontological Innocence asserts that logic is *ontologically* unconditioned; Universal Applicability, that it is *metaphysically* unconditioned; and Cognitive Primacy, that it is *cognitively* unconditioned.

II

I will now have a look at the debate about Ontological Innocence. Boolos defends Ontological Innocence by appealing to intuitions about various English sentences that contain plural quantifiers. Some of these intuitions are directly concerned with ontological commitments. For instance, Boolos denies that the sentence 'There are some Cheerios in the bowl' is committed to anything other than individual Cheerios and a bowl. More controversially, he extends this claim to arbitrary, and thus far more complicated, sentences of PFO. For instance, he denies that the Geach-Kaplan sentence is about anything other than individual critics.

Other of Boolos' intuitions are concerned with the truth of certain sentences, but are taken to have consequences for ontology as well. For instance, Boolos takes it to be intuitively evident that sentences can be true although they are concerned with collections that are too large to form sets. The following sentence provides an example.

(3) There are some sets that are all and only the non-self-membered sets.

Since (3) is true and there is no set of all non-self-membered sets, it's natural to think that the plural construction in (3) is ontologically innocent. However, for this ontological conclusion to follow from the truth of (3), we need to grant Boolos two further assumptions: first, that sets are all the "set-like" entities there are; and second, that we can quantify over absolutely all sets. Unless sets are all the "set-like" things there are, (3) may be interpreted as asserting the existence of a *class* that is too large to form a set. And unless we can quantify over absolutely all sets, we may interpret (3) as asserting the existence of a set that is not in our original domain of quantification. (I will have more to say about this latter approach in the next section.) But both assumptions are controversial: As we will see in the next section, they are at the center of a related controversy. Consequently, this argument has little suasive force. (It does, however, show that Boolos' view has a certain inner coherence.)

Even if one agrees with Boolos' intuition that the plural comprehension axioms are ontologically innocent, one may worry that we will be forced to introduce new ontological commitments when we develop a semantics for the language PFO. To assuage this worry, Boolos develops a semantics for PFO, which, he claims, introduces no new objects.[11] If this is right, it shows that such sentences can be analyzed and interpreted without having to invoke collections or other "set-like" entities to serve as values of the plural variables and expressions. The core idea on which Boolos' proposal is based is to develop the requisite semantic theory in a background language that itself contains plural quantification. This could be our language PFO or, as in Boolos' original exposition, ordinary English slightly supplemented with mathematical terminology. The proposed semantic theory employs a second-order variable 'R' that codes assignments to the second-order variables. Intuitively, $R\langle V, x\rangle$ holds if x belongs to the plurality assigned to the second-order variable 'V'.[12]

I now proceed to the objections to Ontological Innocence. I will focus on the objections due to Charles Parsons, as these seem to me the most developed.[13] Parsons remains unconvinced by Boolos' arguments. First, he disputes Boolos' intuition that sentences containing plural quantifiers say nothing about collections or pluralities.[14] Admittedly, there may not be much of a problem about very simple uses of plural quantification, as in the sentence about the Cheerios in the bowl. In these cases Boolos' claim that no objects are introduced other than those already in the first-order domain seems fairly plausible. But things change when we consider more complicated sentences of PFO that make use of plural cross reference, in particular when these sentences are non-first-orderizable. Consider, for example, the sentence

(4) Whenever there are *some natural numbers* such that 0 is one of *them* and for every natural number n that is one of *them*, $n+1$ is one of them, then every natural number is one of *them*.

It is far from clear, Parsons claims, that when sentences such as (4) are uttered, the utterer cannot be said to have made a claim about *collections* or *pluralities* of natural numbers. It is very natural to regard the occurrences of 'them' as referring to an entity of this sort.

Boolos could respond by calling attention to the Geach-Kaplan sentence, (GK), which, just like (4), contains plural cross reference and is non-first-orderizable.[15] According to Boolos, (GK) doesn't intuitively seem to commit us to any plurality or any other "set-like" entity over and above the individual critics. If correct, this would be a counterexample to Parsons' claim. However, this response has little force against Parsons, who denies Boolos' intuition that (GK) is ontologically non-committal.

Parsons also denies that the semantics Boolos proposes leaves no room for talk about values assigned to the plural variables. Rather, he suggests that X is the value of such a variable just in case it is $\lambda x.R\langle V, x\rangle$ for some R and V; that is, if $\exists R.\exists V.\forall x(Xx \leftrightarrow R\langle V, x\rangle)$.

What are we to make of these arguments and these playings of intuitions against each other? To me, the above discussion suggests that our intuitions about these matters don't carry much weight and that they are insufficient to resolve the dispute about Ontological Innocence. In order to make progress we should look for more theoretical arguments. One way to do this is by examining *the concept of an object* that is at work in this dispute.

Parsons' arguments appear to be premised on what he elsewhere calls "the logical concept of an object."[16] This concept of an object is supposed to provide a systematic account of objecthood and of ontological commitments. The central idea, which goes back to Frege and has since been endorsed by important figures in analytic philosophy such as Carnap, Quine, and Dummett, is that our only general concept of an object is given content only in connection with modern quantificational logic. On this account, the concept of an object is closely related to the concepts of a singular term and of first-order quantification: An object is the sort of thing that singular terms denote and that first-order quantifiers range over. Thus, assume that a sentence S is recognized as true. If t is a sub-expression of S that functions logically as a singular term, then nothing more is needed for there to be an object that t denotes. And similarly, if Q is a sub-expression of S that functions logically as a quantifier, then this suffices for there to be the sort of objects that Q must be taken to range over in order for S to be true. Note that at this point, nothing is said about the *nature* of these objects; in particular, it is not claimed that they are concrete.

Of course, this characterization leaves it open by what standards we are to recognize S as true and how we are to identify sub-expressions of S as singular terms and quantifiers. The proponents of the logical concept of an object agree that these questions should be answered without invoking the general notion of an object that this proposal is intended to explicate.[17] But they disagree about *how* to do this. The most famous proposal is Quine's. Quine suggests that questions about existence be approached in the following way. First, we should formalize the relevant claims and theories in the language of classical first-order logic. In this language the quantifiers and the class of singular terms are clearly delineated. A sentence of this language should then be accepted as true just in case it is implied by a formalization, in the same language, of our best scientific theory of the world.

In his discussion of plural quantification, Parsons invokes an idea "a little different from but complementing Quine's" (Parsons 1990, 299).

The idea is to extend the Quinean suggestion just sketched from the language of classical first-order logic to PFO. As we've seen, PFO contains plural expressions and devices for plural cross-reference, and it allows existential generalization to be applied to these occurrences. The extension of the Quinean suggestion says that, since the existential quantifier means *existence*, nothing more is required to be existentially committed to pluralities.

This argument is particularly strong in the case of languages with *non-distributive* plural predication. Such languages go beyond the expressive power of PFO by allowing predicates and relations (other than the *logical* relation $\prec$) which take plural expressions as arguments and for which predication isn't distributive. (The plural predication '$F(xx)$' is said to be *distributive* just in case it is equivalent to '$\forall x(x \prec xx \leftrightarrow Fx)$'. Examples of distributive and non-distributive plural predications are, respectively, 'The boys ran across the field' and 'The boys lifted the piano'.) Let *PFO* + be the extension of PFO that allows non-distributive plural predication.[18] When plural expressions in this way are allowed to occur as subjects of true non-distributive predications, it is particularly hard not to regard them as standing for entities.[19] Moreover, in PFO+ we can introduce an identity predicate that holds between pluralities: $xx = yy \leftrightarrow \forall u \; (u \prec xx \leftrightarrow u \prec yy)$. This too indicates that reification takes place.[20]

However, since no substantial defense has been given of the logical concept of an object, either in Quine's original form or in Parsons' slight strengthening of it, this argument against the Ontological Innocence of PFO and PFO+ remains somewhat inconclusive. To be conclusive, the argument would have to be buttressed by an independent defense of the logical concept of an object. Although I am sympathetic with this account of the concept of an object, I won't attempt to defend it here.[21] However, even without an independent defense of the logical concept of an object, the argument I have sketched does represent a serious challenge to Boolos. By proposing a clear and theoretically motivated concept of an object from which it follows that plural quantification introduces new ontological commitments, this argument puts the burden of proof on Boolos. To defend his claim that plural quantification is ontologically innocent, Boolos would either have to offer an alternative characterization of objecthood or else give reasons why the logical concept of an object should be rejected. Unfortunately, neither of these responses is worked out in any of Boolos' published writings. Nor do later proponents of Boolos' position develop any such response.

Lacking an alternative account of objecthood, it is no longer so easy to see what is at issue in the dispute about Ontological Innocence. Of course, the "official" answer is that the dispute is about ontology: Parsons affirms and Boolos denies that plural quantification commits one to pluralities. But it is not at all clear what this "official" answer amounts to. I have argued that our intuitions about the ontological commitments of English sentences containing plural quantifiers fail to provide a good handle on the issue. And further, when we reflect on how "thin" the logical concept of an object is from which it follows that plural quantification is committed to pluralities, it becomes even less clear what is at stake in this disagreement. In the absence of a shared account of objecthood, it therefore seems that the disagreement about Ontological Innocence has content only insofar as it bears on other issues that do not belong to "bare" ontology.

III

So I suggest that we turn to the broader question about the logical status of PFO. I will examine this question by considering the use to which Boolos wants to put the theory of plural quantification. Two areas of application come to mind: set theory, and the neo-logicist project suggested by Wright 1983. No doubt, both these areas were very important to Boolos; roughly, they correspond to Parts I and II of his collected papers, *Logic, Logic, and Logic* (Boolos 1998). But as the issues we are interested in arise in a clearer fashion in connection with set theory, and as this application represents the greater hurdle for the view that plural quantification is pure logic, my focus will be on set theory rather than on neo-logicism.[22]

According to Boolos, set theory needs second-order logic for several different reasons. The first reason has to do with the formalization of talk about collections of sets that cannot, at pains of paradox, themselves be sets.[23] Consider, again, sentence (3):

(3) There are some sets that are all and only the non-self-membered sets.

This sentence appears to be perfectly meaningful; indeed, it appears to be true. But it cannot be paraphrased in the standard set-theoretic way because this would involve positing a Russell set, which would immediately lead to paradox.

Presumably, Boolos' interest in sentences of this sort is to a large extent an interest in finding formalizations that accurately represent their logical forms. However, since our current topic is the foundations of mathematics, not linguistics, I see no reason why we should insist on taking these sentences at face value. So long as the same mathematical content is expressed, we should be allowed to paraphrase. And (3) can be paraphrased by means of a first-order sentence, to which it is logically equivalent (in second-order logic), namely

(3′) $\exists x$(x is non-self-membered).[24]

However, there may be other sentences of this sort that cannot be paraphrased in this way. If so, these sentences can be construed as limiting cases of a *second reason* why set theory needs second-order logic, to which I now turn.

In set theory, we sometimes prove results that hold no matter what set theoretic predicate is used to replace some schematic predicate letter. For instance, we know that whenever *F* is a set theoretic predicate, the union of the ordinal numbers that satisfy *F* is well-ordered by the membership relation. Boolos recognizes that in order to express results of this sort, we need a way of quantifying into predicate positions.[25] One way to accommodate such quantification is by adding to our set theory a theory of second-order logic. In fact, it is easily seen that predicative MSOL suffices for this purpose. Another way, which will be described shortly, is by adding a theory of truth.

Thirdly, and most importantly, Boolos thinks that second-order logic is needed in order fully to capture the intended meaning of the axioms of Replacement and Separation.[26]

Having recognized these three reasons why set theory needs second-order logic, Boolos faces a problem. For on the traditional understanding of second-order logic, there must be *entities* for the second-order variables to range over. Two popular and well understood candidates are *sets*—explained, at least in large part, by the so-called iterative conception of set—and *classes*—explained in terms of predication. A third candidate, which is important historically but significantly less well understood, is *Fregean concepts*. The two better understood alternatives are nicely discussed in the work of Charles Parsons. In fact, it appears to have been Parsons' work on these matters that spurred Boolos' interest in plural quantification. I will therefore briefly summarize Parsons' discussion of these two alternatives.

According to Parsons, the need for second-order logic in set theory naturally leads us to a three-step extension of our set theory.[27]

The first step is to add to ZFC a weak theory of classes. According to Parsons, the concept of *class* is based on considerations having to do with predication and quantification into predicate places.[28] It is therefore perfectly suited to accommodate the first two reasons why set theory needs second-order logic. Consider again the statement that whenever *F* is a set theoretic predicate, the union of the ordinal numbers that satisfy *F* is well-ordered by the membership relation. In stating this result, we quantify into a predicate position. Parsons argues that this sort of quantification can very naturally be understood in terms of substitutional quantification over predicates. On this proposal, the original quantified statement is true just in case all the formulas are true that we get by replacing the second-order variable with appropriate predicate expressions, possibly with free first-order variables. This interpretation of the second-order quantifiers can be seen to justify *predicative* second-order logic but not the full impredicative comprehension schema.[29]

There are two different, but equivalent, ways of technically spelling out this extension by a predicative theory of classes. The first way is simply to add to ZFC a predicative theory of classes, where bound class variables are not allowed to occur in instances of the Replacement or Separation schemata. This gives us a weak extension of ZFC known as von Neumann-Bernays-Gödel set theory, or NBG. In fact, NBG can be proven to be conservative over ZFC.[30] The second way is to add to ZFC a satisfaction predicate and axioms that give a recursive characterization of this predicate. Let ZFC′ be the resulting theory, but with the restriction that the satisfaction predicate not be allowed to occur in instances of the Replacement and Separation schemata. Within ZFC′ we can express, by means of Gödelization and the satisfaction-predicate, the substitutional interpretation of the second-order quantifiers that we mentioned above.[31] This allows us to interpret NBG in ZFC′. Conversely, in NBG we can define a satisfaction predicate, which allows us to interpret ZFC′. NBG is therefore equivalent to ZFC′.[32]

The entities over which the second-order variables of NBG range are *classes*. Based on the equivalence of NBG with ZFC′, Parsons argues that these classes can be taken to be *language-constituted*, in the sense that their existence depends only on the existence of the corresponding linguistic predicates.[33]

The concept of *set* is rather different. This concept draws heavily on combinatorial intuitions, which tell us that whenever we have some well-distinguished objects, we can choose some or all of these objects and collect them together to form a set. The familiar iterative conception of set[34] is intended to capture these combinatorial intuitions. Admittedly, the concept of set draws on other intuitions as well.[35] For instance, the axiom of Replacement seems to be motivated in large part by considerations having to do with limitation of size. But the most important respect in which sets differ from classes is by drawing so heavily on combinatorial intuitions.

We now proceed to the second step of Parsons' extension procedure. This step has to do with the third reason Boolos gives why set theory needs second-order logic, namely in order to capture the intended meaning of the axioms of Replacement and Separation. Predicative second-order logic is insufficient for this purpose; to capture *all* the intended instances of the Replacement and Separation schemata, we need the full impredicative class comprehension schema. We are thus led beyond NBG to a class theory with full impredicative class comprehension. This theory is known as Morse-Kelley set theory, or MK. MK is much stronger than NBG, both in terms of logical strength and in its ontological commitments. Because of its use of impredicative reasoning, MK cannot be justified by considerations having to do with predication and quantification into predicate positions. Rather, to justify the impredicative class comprehension schema, we need to appeal to the concept of an arbitrary sub-collection, which draws on certain combinatorial intuitions.

At this point Parsons suggests, somewhat surprisingly, that a third step of the extension procedure be carried out. This step is to "nominalize" the values of the second-order variables of MK; that is, to regard the second-order entities over which these variables range as (first-order) objects. This nominalization may be carried out by first construing the second-order theory MK as a two-sorted first-order theory and then grouping these sorts together to form a one-sorted first-order theory. This step amounts to treating the domain of our original interpretation of ZFC as itself a set. (In the presence of the separation schema of ZFC, this guarantees that there are sets corresponding to all the classes of MK.) Thus, when this final step is carried out, we let our quantifiers range over a domain of sets that is larger than the one we started out with.

Why should we make this third step? Parsons' argument has to do with the strength of the considerations that are needed to justify the second step. Already that step, he claims, makes essential use of the sorts of combinatorial intuitions that give content to the concept of set.[36] So set theoretic ideas are introduced already at the second step. There is thus nothing that prevents us from making the final third step. In fact, making this step is extremely natural. It amounts to little more than making explicit ideas that were present already at the second step.

If Parsons is right about this, a rather surprising consequence follows: It is impossible ever to quantify over absolutely all sets. For whenever a domain is specified for the first-order quantifiers of our set theory, we can carry out this three-step extension procedure, which will lead us to accept an even larger domain of sets.

To summarize and fix our terminology, we have seen that, according to Parsons, the application of second-order logic to set theory commits us to two somewhat surprising claims:

> *Ontological Proliferation*. There is more than one ontological category of "set-like" entities: in addition to *sets* there are *classes*.
>
> *Inexhaustibility*. It is impossible ever to quantify over absolutely all sets.

Ontological Proliferation occurs already at the first step of the extension procedure, whereas Inexhaustibility, which is by far the most surprising of these two claims, occurs only at the third step.

George Boolos finds both of these claims very unappealing. He holds that set theory was supposed to be a theory of all the "set-like" things there are, and he finds it deeply problematic that we shouldn't be able to quantify over absolutely all sets.[37] However, since Boolos believes that set theory needs second-order logic, he is faced with the problem of accommodating this need while still denying Ontological Proliferation and Inexhaustibility. Because he denies Ontological Proliferation, Boolos cannot let the second-order variables range over classes. Nor can he let them range over sets; for this conflicts with his denial of Inexhaustibility. (By the negation of Inexhaustibility, we may let the first-order variables of our set theory range over absolutely all sets. But since the domain of all sets is not a set, this second-order variable in the comprehension axiom '$\exists X.\forall x.(Xx \leftrightarrow x = x)$' cannot be assigned a set as its value.) Thus, neither of the two better understood interpretations of second-order logic is available to Boolos. So he seems to have gotten himself into a very awkward position.

At this critical point Boolos appeals to his theory of plural quantification. According to Boolos, plural quantification allows us to apply MSOL to set theory without introducing any new entities or expanding our universe of sets. Recall the three claims that I suggested, in Section I, as a partial analysis of the claim that the theory of PFO is pure logic: Ontological Innocence, Universal Applicability, and Cognitive Primacy. If plural quantification is logical in the sense laid down by these claims, then Boolos will have solved the problem with which he is confronted. For by Universal Applicability, the theory of PFO may be applied to set theory; by Ontological Innocence, this application introduces no new ontological commitments; and by Cognitive Primacy, plural quantification can be understood without smuggling in any set theoretic ideas through the back door.

Conversely, if plural quantification is to offer Boolos a way out of his awkward position, something very close to these three claims will be needed. Obviously, all three claims must hold, if not universally, then at least of set theory. Firstly, plural quantification must be applicable to set theory. Secondly, thus applied, it must be ontologically innocent. And thirdly, it must be cognitively prior to set theory. However, I believe it is possible to sharpen this result. If plural quantification can be innocently applied to set theory, and if Boolos is right that sets are all the "set-like" entities there are, then it is extremely hard to see how the original claims of Ontological Innocence and Universal Applicability could fail to hold. For if sets are all the "set-like" entities there are, then presumably any domain whatsoever can be modeled in the domain of sets. This means that, for any structure, plural quantification can be innocently applied to an *isomorphic copy* of this structure. This makes it extremely hard to see how plural quantification could fail to be innocently applicable to the original structure as well. Finally, with regard to Cognitive Primacy, it seems that the best way to defend the cognitive primacy of plural quantification over set theory is by defending the original stronger claim that plural quantification is cognitively unconditioned.

I conclude that the logicality of the theory of plural quantification stands or falls with the defensibility of Boolos' position in this controversy with Parsons.[38]

IV

I will now attempt to show that Boolos' position in this controversy is indefensible. In rough outline, my argument goes as follows. I argue that the considerations that allow us to add the theory of plural quantification to first-order theories are strong enough to support *iterated* extensions of this sort as well: These considerations allow us to add higher and higher levels of plural quantification. This represents an instability in Boolos' position, which drives him towards a theory that exhibits a phenomenon of the same general kind as that which is involved in Parsons' Inexhaustibility.

Recall from Section I that the only serious challenge to the view that the theory of plural quantification is pure logic comes from the plural comprehension axioms. It should therefore come as no surprise that these axioms play a crucial role in my argument that the extension Boolos proposes is unstable.

Adding the theory of plural quantification to an interpreted first-order theory involves adopting the plural comprehension axioms, applied to the domain of this theory. What justifies us in adopting these axioms? Because we want the impredicative plural comprehension axioms as well as the predicative ones, it's not enough to be justified in taking there to be pluralities[39] corresponding to all *predicative* substitution instances for the plural variables; that is, in taking there to be pluralities corresponding to all expressions of the form

a_1 and ... and a_m and the φs,

where m is a natural number, the a_i's are singular terms, and φ contains no bound plural variables. Rather, what we need to justify is that there are pluralities corresponding to *all* expressions of the form 'the φs', even where φ contains bound plural variables. But in order to do this, we must understand what these bound plural variables range over. This means that we must understand the notion of a *determinate range of arbitrary sub-pluralities* of the original domain.[40]

The notion of a determinate range of arbitrary sub-pluralities can hardly be taken to be primitive and unanalyzable. So we need an account of the considerations that give content to it. The need for an account of this notion becomes particularly acute when we want to apply the plural comprehension axioms to the domain of higher set theory. For applied to this domain, the notion of a determinate range of arbitrary sub-pluralities becomes extremely complicated and abstract. In fact, the notion of an arbitrary sub*set*, which is closely related to but weaker than that of an arbitrary sub-*plurality*, is one of the most difficult and problematic concepts of all of set theory, rivaled in this respect only by the concept of an arbitrary ordinal number. For instance, if we understand the notion of an arbitrary subset, we understand all the concepts that are needed to express the Continuum Hypothesis. So it would be highly unreasonable to regard this notion as unanalyzable.

I believe the considerations that give content to the notion of a determinate range of arbitrary sub-pluralities belong to combinatorics and to set theory. These considerations give a rather good understanding of the range of arbitrary sub-pluralities of the sorts of small, finite collections that most ordinary English discourse is concerned with. Presumably, this understanding is based on combinatorial operations that we can perform on these collections; for instance, going through the items one by one, selecting some and rejecting others. And to whatever extent we understand *infinitary* uses of the notion of an arbitrary sub-plurality, we do so in virtue of the same sorts of combinatorial and set theoretic intuitions that give content to this notion when it is applied to finite domains. To get a handle on the notion of an arbitrary sub-plurality of a given infinite collection, we rely on our insights from the finite case and attempt to extrapolate to infinite cases.[41]

On this account, the notion of a determinate range of arbitrary sub-pluralities requires, in order to be understood, a prior understanding of combinatorics and possibly even of set theory. But given this dependence, nothing debars us from collecting together these pluralities to form ‘‘higher pluralities’’—first pluralities of pluralities and then pluralities of even higher levels. For the iterability of the operation of forming collections is essential to combinatorics as well as to set theory. So on this way of giving content to the notion of a range of arbitrary sub-pluralities, we are naturally led to form higher pluralities as well. Clearly, when we quantify over these pluralities, we get theories that go far beyond PFO. This means that Boolos' position is unstable: adding PFO to set theory naturally leads to even stronger extensions of set theory.

How could Boolos respond to this argument? I will now consider two attempts to defend the stability of Boolos' position. The first attempt is based on the structure of natural language, and I think it represents what would have been Boolos' own response to the argument I just gave. What is needed for Boolos' position to be stable is some considerations that justify the addition to every first-order theory of one layer of plural quantification but which fail to justify further extensions. Now, if plural quantification is a primitive linguistic device that is understood already in virtue of understanding English, this appears be exactly what Boolos needs. For as speakers of English, we seem to understand one level of plural quantification over any domain. And English seems not to contain any higher plural constructions. Hence, it appears that no further extensions are warranted and that Boolos' position is stable after all.

This defense of Boolos' position can also be cast as an objection to my argument. Consider the following two claims that I made in the course of my argument: first, that the notion of an arbitrary sub-plurality that is employed in connection with ordinary English plural quantification gets its content from combinatorics and possibly even from set theory; and second, that the iterability of the operation of forming collections or pluralities is essential to both these disciplines. From these claims one would expect that English too would contain devices for higher plural quantification. However, since English contains no such devices, doubt seems to be cast on my claims.

I think there is a perfectly good answer to this objection. The reason why English contains no separate devices for higher plural quantification is that a language that does will be cumbersome and unpractical, and that ordinary English, just as it is, offers better ways of expressing essentially the same content (by which I mean the same content *modulo* concerns about ontological commitments). Instead of having a stock of separate syntactical devices to handle second-order plural quantification and maybe even more devices to handle plural quantification of higher orders, English allows essentially the same content to be expressed by *singularizing* the first-order pluralities. Assume, for instance, that some Cheerios are arranged thus: oo oo oo. In English we can describe this by saying that there are three *pairs* of Cheerios arranged colinearly. In talking about pairs of Cheerios, we singularize the two-membered pluralities. The advantage of this singularization is that it allows us to employ the familiar and powerful apparatus that English provides for talking about individual objects.[42] Clearly, this is much easier than operating with *separate* natural language devices that would allow us to express, without such singularization, the higher-order plural statement that can be expressed (in an obvious extension of our language PFO) thus:

> $\exists xxx\, \forall xx\, (xx \prec xxx \leftrightarrow xx = aa \vee xx = bb \vee xx = cc)$ & *aa* are two Cheerios & *bb* are two Cheerios & *cc* are two Cheerios & *aa* and *bb* and *cc* are arranged on a line.

It follows that our question—whether the extension that Boolos proposes is stable—cannot be settled simply by appealing to the structure of English. For the fact that English contains no higher-level plural constructions has an *independent* explanation and hence does not conflict with my claim that the considerations that give content to plural quantification allow iterated applications of Boolos' extension as well.

I now proceed to the second attempt to defend the stability of Boolos' position. This defense is of an ontological nature. It is based on the idea that only *things* can be collected together. If this idea is right, and if Boolos avoids reifying pluralities, then there will simply be no *things* available to be collected together to form higher pluralities. But this defense too is unconvincing. There is no obstacle to iterating the combinatorial considerations that give content to our talk about arbitrary sub-pluralities; in particular, combinatorics has no ontological qualms about collecting together first-order pluralities so as to form higher pluralities. For instance, from the point of view of combinatorics, it is no more problematic to arrange individual Cheerios in the following way: oo oo oo than it is to arrange them as: oooooo, although the former arrangement is most informatively described as three pairs of Cheerios—which is a higher-order plurality—whereas the latter arrangement is a mere first-order plurality based on the same six Cheerios. To whatever extent the more complex arrangement involves additional ontological commitments, these commitments pose no problem to combinatorics.

So my conclusion that Boolos' position is unstable remains unchanged: There is no conception of plural quantification that allows us to add impredicative PFO to ZFC set theory without naturally leading to further extensions as well. This means that if Boolos wants to apply the theory of plural quantification to set theory, he will have to accept higher plural quantification as well. This leads to a stratified theory of higher pluralities. When we develop this theory up to the level of some ordinal number α, the resulting theory will be isomorphic with impredicative simple type theory of order α, in the sense of being equi-interpretable with it.[43]

Finally, let's consider the question of Inexhaustibility. We have seen that Boolos is driven to a theory where it is always possible to add new layers of even higher pluralities. It follows from Cantor's theorem[44] that each new layer of pluralities gives us a domain that is larger than the previous one. Clearly, if we were to singularize these higher pluralities—that is, if we were to treat them as sets—we would surrender all hope of being able to quantify over absolutely all sets: each new layer of pluralities would then lead to a greater domain of sets. However, even if we decide not to singularize the higher pluralities, the situation won't be essentially different. For at no level of this theory will we be able to quantify over all the pluralities there are: There will always be higher levels. This situation exhibits a limitation of the same general kind as that which led Parsons to deny that we can quantify over absolutely all sets.

In Section IV, I argued that the claim that plural quantification is pure logic stands or falls with the defensibility of Boolos' position in the controversy with Parsons. Now, I have argued that Boolos' position in this controversy is indefensible. So if my arguments are correct, it follows that the theory of plural quantification has no right to the title "logic."

V

Having reached this negative conclusion about the logical status of PFO, I will, in closing, consider where this leave us with regard to the question of Ontological Innocence. In Section II, I discussed two attempts to get a handle on this question. First, I considered and rejected the most common approach, based on bandying intuitions about various English sentences. Then, I considered Parsons' slight extension of Quine's logical concept of an object. Although I found this proposal to be quite congenial, I made no attempt to defend it. However, reflecting on the extremely "thin" nature of the pluralities to which plural quantification would be committed were we to adopt this concept of an object, I found it to be very hard to make sense of the "bare" ontological question whether or not there are such pluralities.

I therefore suggested that insofar as the question of Ontological Innocence has any substantive content at all, this content must have to do with questions that are not concerned with "bare" ontology.

I would now like to return to this suggestion and elaborate on it. According to this suggestion, the substantive content of the question of Ontological Innocence is a matter of what the truth of the plural comprehension axioms *consists in*. If this suggestion is accepted, there will be a close connection between the argument of the last section that PFO isn't logical and the question of Ontological Innocence. According to that argument, the impredicative plural comprehension axioms fail to be logical because they depend too heavily on combinatorial and set theoretic considerations. Their truth, in other words, consists in much the same as does the truth of the corresponding *set theoretic* comprehension axioms. The present suggestion therefore implies that *both* sorts of comprehension axioms carry ontological commitments.

One may, of course, reject the suggestion that Ontological Innocence is a matter of what the truth of the plural comprehension axioms consists in. I won't attempt any systematic defense of this suggestion here. Rather, I will just remark that if one does reject this suggestion, one will be left with a notion of ontological commitment that looks rather unattractive. For one thing, this will be a notion of ontological commitment that seems to resist all serious attempts at explication. For another, this notion of ontological commitment will lead to the conclusion that two theories—in our case, set theory with Inexhaustibility and set theory coupled with an inexhaustible hierarchy of higher plural quantification—can be logically equivalent yet still have radically different ontological commitments. But if this is so, there is reason to wonder why this notion of ontological commitment should be so important to the philosophy of mathematics in the first place.

Notes

* This paper grew out of a series of discussions with Agustín Rayo in the spring of 2000. I learnt a great deal from these discussions, and I am grateful to Agustín for questions and comments that have much improved this paper. Thanks also to Matti Eklund, Michael Glanzberg, Warren Goldfarb, the participants in a joint Harvard-MIT discussion group, an anonymous referee for *Noûs*, and especially to Charles Parsons for valuable comments on earlier versions of this paper.

[1] See Boolos 1984 and 1985. *Monadic* second-order logic is second-order logic whose only second-order variables are monadic. I will refer to both the language and theory of monadic second-order logic as *MSOL*. This ambiguity is innocent; it will always be clear from the context whether I mean the language or the theory.

[2] See Resnik 1988, Parsons 1990, and Hazen 1993.

[3] For a representative sample of readings of Boolos' proposal that emphasize its potential for ontological parsimony, see Burgess and Rosen 1997, II.C.0 and II.C.1.b; Dorr and Rosen forthcoming; Hellman 1994 and 1996; and Lewis 1991 and 1993.

[4] To make this inference, first apply EI to '$\exists x(x = t)$' to get '$x = t$'. From '$x = t$' and '$\varphi(t)$' we derive '$\varphi(x)$' by the laws of identity. And from '$\varphi(x)$' we finally infer '$\exists x.\varphi(x)$' by EG.

[5] My presentation of this language follows Rayo 2002.

[6] For a proof of this latter claim, see Boolos 1998, 57. (Boolos attributes this proof to David Kaplan.)

[7] Strictly speaking, this reduction of the question about the logical status of MSOL to the corresponding question about PFO presupposes that the specified translation preserve meanings well enough to transfer logical status. But if someone is worried about this, she would be advised to accept the theory of plural quantification directly and simply use *this* theory, rather than MSOL, to accommodate whatever needs she has for second-order quantification. Either way we have to address the question about the logical status of the theory of plural quantification.

[8] See e.g., Boolos 1998, 167.

[9] Before Boolos discovered the interpretation of MSOL in terms of plural quantification, he was worried that this would deprive second-order logic of its right to be called "logic." See Boolos 1998, 42–5.

[10] Clearly, this appeal to what is "extra-logical" disqualifies my analysis from being a *definition* of logicality. However, all I aim for is a rough characterization. A lot of concepts, such as that of *set*, are widely agreed to be extra-logical.

[11] Strictly speaking, the semantics Boolos develops is for a fragment of English. But it extends in an obvious way to our language PFO.

[12] For technical details, see Boolos 1998, 79–83. Note that this presupposes that for every two objects we can form, or in some way code for, their ordered pair.

[13] See in particular Parsons 1990, 296–300.

[14] See Resnik 1988 for a similar objection, developed at somewhat greater length.

[15] See note 6.

[16] For a discussion of the logical concept of an object, see Parsons 1982.

[17] This restriction would be violated if, for instance, we characterized the notion of a singular term by stipulating that '*t*' is a singular term just in case there is an object that it denotes.

[18] For a discussion of the language PFO+, see Rayo 2002.

[19] See Parsons 1990, 299, in particular footnote 64.

[20] In connection with this discussion of PFO+ it should be observed that the semantic theory Boolos proposes for PFO makes use of non-distributive plural predication, namely in the first argument place of the satisfaction relation $\mathrm{Sat}(R, s, \varphi)$. This means that his semantic theory in fact requires PFO+.

[21] I do, however, develop such a defense in work in progress.

[22] The application of plural quantification to the neo-logicist project may allow us to connect the question about the logical status of the theory of plural quantification with the question about the logical status of the weak arithmetical theory consisting of PFO and the right-to-left half of Hume's Principle, $\approx_x(Fx,Gx) \rightarrow \mathrm{N}x.Fx = \mathrm{N}x.Gx$ (which may with some plausibility be regarded as logical), or with the question about the relative logicality of second-order Frege Arithmetic over first-order Frege Arithmetic. However, it seems to me that these questionss are no better understood than our original question.

[23] See Boolos 1998, 64–5.

[24] With the non-vacuity condition introduced earlier, (3) can be formalized as $\exists X.[\forall x.(Xx \leftrightarrow x \notin x)\ \&\ \exists x.Xx]$. By instantiating quantifiers, one easily deduces $\exists x.(x \notin x)$. Conversely, from $\exists x.(x \notin x)$ and the comprehension axiom $\exists X.\forall x.(Xx \leftrightarrow x \notin x)$, we can prove (3) (formalized as above).

[25] See Boolos 1998, essay 2, in particular 33–5.

[26] See Boolos 1998, 65.

[27] For details, see Parsons 1983, essays 3 and 8.

[28] See Parsons 1983, essays 2, 3, and 8.

[29] See e.g., Parsons 1983, 69.

[30] In NBG we disallow bound second-order variables from occurring in instances of the Replacement and Separation schemata. If we lifts this restriction, we get a somewhat

stronger extension of ZFC, which I will call NBG^+. The satisfaction predicate that we can define in NBG^+ is allowed to occur in instances of the Replacement and Separation schemata. This allows us to prove that all the theorems of ZFC are true, and hence that ZFC is consistent. So by Gödel's second incompleteness theorem it follows that NBG^+ is non-conservative over ZFC.

[31] See Parsons 1983, 213–4.

[32] Here's another fact. Let ZFC″ be like ZFC′ but with no restriction on the occurrences of the satisfaction predicate in instances of axiom schemata. Then ZFC″ is equivalent to NBG^+ (from note 30).

[33] See Parsons 1983, 66 and 285.

[34] See e.g., Boolos 1998, essay 2, and Parsons 1983, essay 10.

[35] This is argued in Parsons 1983, essay 10, and (in a stronger form) in Parsons 1995.

[36] See Parsons 1983, 69 and 217–8.

[37] See, respectively, Boolos 1998, 35, 43, and 66; and 30–3.

[38] As mentioned in the previous paragraph, a minor qualification is needed about the claim that Boolos' position *stands* with the logicality of plural quantification. However, since this qualification is irrelevant to the remainder of my argument, it will henceforth be suppressed.

[39] For the sake of clarity and succinctness I will help myself to platonistic locutions such as "plurality." However, the following argument is meant to go through whether or not pluralities are construed as entities.

[40] The claim we must be able to understand can be given the following ontologically innocent formulation: Whenever some things *xx* form a domain, then it is determined what things *yy*, subject to the condition $\forall y\ (y \prec yy \rightarrow y \prec xx)$, can be values of our plural variables.

[41] For a classical exposition of this idea, see Bernays 1935.

[42] This singularization is closely related to the wide-spread tendency in English to nominalize various syntactical categories in order to facilitate cross-reference and generalizations. Consider, for instance, the sentence 'Sgt. Smith is brave, strong, and enduring; in fact, he seems to have *all the qualities* of a good soldier'. In this sentence, the desired generalization is effected by means of ordinary first-order quantification over qualities.

[43] This argument that Boolos' project leads to a theory of higher plural quantification proceeds *from below* in the sense that it shows that any conception of plural quantification that allows an impredicative plural comprehension schema to be applied to the domain of set theory leads to higher plural quantification as well. There is another argument with the same conclusion that proceeds *from above*. This argument goes as follows. Boolos wants to have the logical and conceptual resources to express the central metalogical concepts, such as validity, satisfiability, and consequence. But in order to express these concepts, he is led through a series of languages with stronger and stronger expressive powers. We have already seen that in order to give a semantics for the language PFO, and in this way be able to define PFO-validity, we need the stronger language PFO+. Further, to give the semantics for PFO+, we need to be able to talk about interpretations of the *plural predicates* of PFO+, that is, the predicates that take first-order pluralities as arguments. Cardinality considerations show that this cannot be done in the language PFO+: there are more such assignments than there are pluralities of objects from the first-order domain. (See Rayo and Uzquiano 1999 for further discussion.) But if we admit second-level pluralities (pluralities of pluralities of individual objects), this can be done in a natural way by assigning to each plural predicate a second-order plurality. Further, since this pattern repeats itself up through the hierarchy of higher plural quantification, no finite level provides a natural stopping point.

[44] Strictly speaking, since the domain doesn't form a set, we need a slight generalization of Cantor's theorem known as *Bernay's Principle*, which states that a domain cannot be mapped onto the collection of pluralities that can be formed from this domain. For a discussion, see Rayo 2002, Section 4.

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