Structuralism and the Notion of Dependence

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The notion of dependence figures prominently in many recent discussions of non-eliminative mathematical structuralism. Supporters of this brand of structuralism often argue that mathematical objects from one and the same structure depend on each other and on the structure to which they belong. Their opponents often disagree, arguing instead that there cannot be any such dependence.

This paper has two goals. The first goal is to show that the structuralists' claims about dependence are more significant to their view than is generally recognized. I argue that these dependence claims play an essential role in the most interesting and plausible characterization of this brand of structuralism. The second goal is to defend a compromise view concerning the dependence relations that obtain between mathematical objects. Two extreme views have tended to dominate the debate, namely the view that *all* mathematical objects depend on the structures to which they belong and the view that *none* do. I present counterexamples to each of these extreme views. I defend instead a compromise view according to which the structuralists are right about many kinds of mathematical objects (roughly, the algebraic ones), whereas the anti-structuralists are right about others (in particular, the sets). I end with some remarks about how to understand the crucial notion of dependence, which despite being at the heart of the debate is rarely examined in any detail.

1 What is non-eliminative mathematical structuralism?

Let's begin by getting clear on the brand of structuralism that will occupy us. The first thing to note is that this brand is a version of *mathematical* structuralism. Very roughly, mathematical structuralism is the view that pure mathematics is the investigation of abstract structures and that all that matters to mathematics are purely structural properties of objects. Mathematical structuralism is thus a *local* form of structuralism, restricted to the field of mathematics. This contrasts with various *global* forms of structuralism, which attempt to say something about objects or our cognitive relation to them quite generally.[1] It also contrasts with other local forms of structuralism, for instance structuralism about the physical world.[2] Mathematical structuralism is distinct from and independent of these other forms of global and local structuralism.

The next important distinction is between *eliminative* and *non-eliminative* versions of mathematical structuralism. It is perhaps easiest to begin by describing the non-eliminative version. This version of mathematical structuralism takes structuralism to be a fundamental insight about the nature of mathematical objects, namely that these objects are really just positions in abstract mathematical structures. The natural number 2, for instance, is just the second (or on some approaches: the third) position in the abstract structure instantiated by all systems of objects satisfying the second-order Dedekind-Peano axioms.

The eliminative versions of mathematical structuralism are unified mostly by their opposition to the non-eliminative version just outlined. The eliminative versions deny that there are abstract mathematical structures and that the nature of mathematical objects is exhausted by their being positions in such structures. We can distinguish between three versions of eliminative structuralism. *Deductivist structuralism* avoids both ontological commitment to mathematical objects and all use of modal vocabulary. It interprets mathematics as the formulation of various (mostly categorical) theories to describe various kinds of concrete structures, and as the study of what holds in all models of each of these theories.[3] *Modal structuralism* lifts the deductivists' ban on modal notions. It interprets mathematics as asserting that it is possible for various theories to have concrete models, and as studying what necessarily holds in all such models.[4] Finally, *set-theoretic structuralism* rejects the deductivists' nominalism in favor of a background theory of sets. It then takes mathematics to be the study of various structures realized among the sets. This is often what mathematicians have in mind when they talk about structuralism.

In what follows my sole concern will be with non-eliminative mathematical structuralism. I will therefore reserve the word 'Structuralism' for this view.

A fundamental question that confronts Structuralism is how this view differs from more traditional platonist views of mathematics. After all, both types of view are ontologically committed to abstract mathematical objects, and both agree that these objects enter into larger structures. An initial Structuralist response is that on their view, mathematical objects

[1]See (Quine, 1992) for an example of a semantic form of global structuralism.

[2]See (Ladyman, 1998) for an example of structuralism about the physical world.

[3]Canonical examples are (Russell, 1903) and (Putnam, 1967b).

[4]An early example is (Putnam, 1967a). The most developed version is (Hellman, 1989).

as *nothing more than* positions in abstract structures, whereas on traditional platonist views, mathematical objects have a richer nature than this. But although suggestive, this initial response needs to be the spelled out. It is of limited use to be told that mathematical objects are identical with positions in abstract structures before an account has been given of the nature of these positions, in particular of how they differ from mathematical objects as conceived by the traditional platonist.

Fortunately, Structuralists make a variety of more substantive claims about how their view differs from more traditional platonist views. Two main claims stand out. Firstly, what we may call the *Incompleteness Claim* says that mathematical objects are incomplete in the sense that they have no "internal nature" and no non-structural properties. The following passage from a seminal early article on Structuralism gives a typical expression of this claim.[5]

> In mathematics, I claim, we do not have objects with an 'internal' composition arranged in structures, we have only structures. The objects of mathematics [...] are structureless points or positions in structures. As positions in structures, they have no identity or features outside a structure. ((Resnik, 1981), p. 530)

Secondly, what we may call the *Dependence Claim* says that mathematical objects from one structure are dependent on each other and/or on the structure to which they belong. This claim is meant to conflict with a more traditional platonist view, which ascribes a greater degree of independence to mathematical objects. The following passage provides a good example of this claim.

> The number 2 is no more and no less than the second position in the natural number structure; and 6 is the sixth position. Neither of them has any independence from the structure in which they are positions, and as positions in this structure, neither number is independent of the other. ((Shapiro, 2000), p. 258)

2 The Incompleteness Claim

As mentioned in the introduction, my main concern in this paper will be with the Dependence Claim. But before I embark on any attempt to analyze and assess the Dependence Claim, it makes sense to pause and discuss whether the claim really matters. What if the claim is just some throw-away remark, intended to carry little or no weight?

[5]This passage is quoted approvingly in a large number of defenses of Structuralism; for instance (Parsons, 1990), p. 303 and (Shapiro, 1997), p. 75.

There are at least two reasons why the Dependence Claim deserves to be taken seriously. The most obvious reason is simply that the Dependence Claim is an interesting metaphysical claim, which almost all Structuralists (explicitly or implicitly) commit themselves to. This will be discussed in the next section. A less obvious reason, which will be our concern in this section, is that the Incompleteness Claim is deeply problematic. Although I won't attempt any conclusive assessment of these problems, I will argue that they are serious enough to make it very unwise for Structuralists not to take the Dependence Claim seriously.

The literature on Structuralism is replete with statements of various forms of the Incompleteness Claim. Some of these statements have to do with there being no 'fact of the matter' about cross-category identity statements, such as whether the natural number 2 is identical to a certain set, or to Julius Caesar.

> Mathematical objects are incomplete in the sense that we have no answers within or without mathematics to questions of whether the objects one mathematical theory discusses are identical to those another treats. ((Resnik, 1997), p. 90)[6]

> [I]t makes no sense to pursue the identity between a place in the natural-number structure and some other object, expecting there to be a fact of the matter. Identity between natural numbers is determinate; identity between numbers and other sorts of objets is not. ((Shapiro, 1997), p. 79)[7]

For this reason mathematical objects are often said to be 'incomplete', somewhat like fictional objects.[8] Other statements claim that the objects of mathematics are structureless points with no internal nature.

> Mathematics is concerned with structures involving mathematical objects and not with the 'internal' nature of the objects themselves. ((Resnik, 1981), p. 529)

> [There] is no more to the individual numbers "in themselves" than the relations they bear to each other" ((Shapiro, 1997), p. 73)

> The idea behind the structuralist view of mathematical objects is that such objects have no more of a 'nature' than is given by the basic relations of a structure to which they belong. ((Parsons, 2004), p. 57)

[6] See also (Resnik, 1997), p. 210.

[7] See also (Shapiro, 1997), pp. 80-81 and 258ff.

[8] See for instance (Parsons, 7980), Section II; (Parsons, 1990), p. 334-5; and (Resnik, 1997), pp. 90 and 211.

An examination of these quotes show that the Incompleteness Claim consists of two *prima facie* different strains. According to the first strain, mathematical objects possess no *non-structural properties*. According to the second strain, such objects have no *internal composition* and more generally no *intrinsic properties*. I will refer to these two strains as respectively *NS-Incompleteness* and *I-Incompleteness*. Depending on what the relation is between the non-structural and the intrinsic properties of mathematical objects, these two strains may and may not be extensionally equivalent. But since the strains are certainly intensionally different, I will discuss them separately.

Let's now attempt to analyze these two strains. When NS-Incompleteness denies that mathematical objects have any non-structural properties, what exactly is meant by a "structural property"? A useful answer to this question can be extracted from the work of Shapiro. Shapiro uses the notion of a *system* to refer to "a collection of objects with certain relations" ((Shapiro, 1997), p. 73). Following Shapiro, I will use a separate kind of upper-case variable to range over "collections" of objects. For present purposes we do not need to take a stand on how these variables (and the quantifiers binding them) are to be interpreted. (The options include pluralities, sets, classes, and properties.) A system in Shapiro's sense is then an ordered $(n+1)$-tuple consisting of a domain D and relations on this domain $R_1, \ldots, R_n$. (There is no need to have separate variables for designated objects, as these can if necessary be represented by means of the relations by using, for each such object, a one-place relation uniquely true of it.)

Next Shapiro introduces the notion of a *structure* as "the abstract form of a system, highlighting the interrelationships among the objects, and ignoring any features of them that do not affect how they relate to other objects in the system" (p. 74). We may refer to this process as *Dedekind-abstraction*, in honor of one of its earliest and most important defenders.[9] Although Shapiro's characterization of Dedekind-abstraction relies heavily on epistemological language, it is clear that the process itself is supposed to be a metaphysical one.[10] A *structural property* can now be characterized as a property that can be arrived at through this process of abstraction; or equivalently, a property that is shared by every system that instantiates the structure in question. The NS-Incompleteness Claim then says that the only properties that mathematical objects possess are the ones that are structural in this sense.

However, the NS-Incompleteness Claim, stated this boldly, is subject to a variety of different counterexamples.[11] For instance, the number 8 has the property of being my favorite

[9]This label is used in (Parsons, 1990) (who attributes it to William Tait).

[10]A thoroughly metaphysical analysis of Dedekind-abstraction will be proposed in Section 6.

[11]See (Burgess, 1999), p. 286; (Linnebo, 2003), pp. 97-9; and (MacBride, 2005), pp. 583-584.

number. It also has the property of being the number of books on one of my shelves. And it has non-structural properties such as being abstract and being a natural number. In fact, the property of being abstract seems to be a very important property of natural numbers. A student of arithmetic who has not realized that numbers are abstract is missing something very fundamental. Finally, as John Burgess observes, it is doubtful that the NS-Incompleteness Claim can even be coherently stated. For the property of having only non-structural properties isn't itself a structural property, as it isn't shared by all systems that instantiate the structure in question.[12]

Because of these problems, at least one important Structuralist, Stewart Shapiro, has recently backed off from most aspects of the Incompleteness Claim. Shapiro now argues that mathematical objects from distinct structures are always (determinately and objectively) distinct; and he comes close to accepting that mathematical objects possess non-structural properties such as being abstract.[13]

Can the idea behind the NS-Incompleteness Claim be salvaged by being given a more modest formulation?[14] Instead of claiming that absolutely all properties of mathematical objects are purely structural, perhaps the Structuralists can restrict the claim to some important kind of properties. But what could this important kind of properties be? Given that abstractness is presumably a necessary property of mathematical objects, it would not suffice to restrict the claim to necessary properties. The only restriction that might work, it seems, would be to those properties of a mathematical object that *matter for its identity.* But this would transform the NS-Incompleteness Claim to something close to the Dependence Claim. For instance, the claim might then be that the identity of a mathematical object is grounded in its structural properties, and that the object in this sense depends upon the structure to which it belongs.

Turning now to the I-Incompleteness Claim, we begin by observing that any "internal composition" of an object would have to be reflected in its intrinsic properties. It is therefore sufficient to consider the claim that mathematical objects have no intrinsic properties. But this claim too is problematic. Here the problem is to make sense of what it would mean for a mathematical object to have intrinsic properties. An intrinsic property is supposed to be a property that an object has solely in virtue of the way it is and not in virtue of the relations it bears to the rest of reality. In the literature there are two main ways of spelling

[12]See (Burgess, 1999), p. 286.

[13]See (Shapiro, 2006a), pp. 121-131.

[14]See (Shapiro, 2006a), p. 115.

out the notion of an intrinsic property.[15] On the first analysis, a property is intrinsic to an object just in case it is shared by every *duplicate* of the object. But this analysis is of little use in connection with mathematics because it is hard to make sense of the notion of a duplicate of a mathematical object. Mathematical objects simply aren't the kind of things we ordinarily think of as having duplicates. (And if a mathematical object has no duplicates except itself, then *all* of its properties will count as intrinsic, in diametrical opposition to the I-Incompleteness Claim.) But let's temporarily waive this worry and ask what would be required of a duplicate relation for it to validate the I-Incompleteness Claim.[16] The answer is that each mathematical object would have to have duplicates which are so diverse that they don't share even a single property. This would mean that the duplicates of, say, a natural number would have to include non-numbers and even objects that aren't abstract! I conclude that on this first approach to intrinsicness we would be forced to operate with a duplicate relation that is both poorly understood and that would clash with the few intuitions we may have about it.

On the second analysis, a property is intrinsic to an object just in case the object would have this property even if the rest of the universe were removed or disregarded. This is somewhat more promising. For on this analysis, the I-Incompleteness Claim can be understood as the claim that no mathematical object can possess properties completely on its own but rather that a property can meaningfully be ascribed to a mathematical object only when this object is regarded as part of the structure to which it belongs. However, thus understood the I-Incompleteness Claim comes to little more than the Dependence Claim. For to claim that an object cannot be considered in isolation from its structure, and that it cannot possess any properties except as a component of this structure, is to make a claim about how the object *depends upon* this structure.

To sum up, we have seen that the first strain of the Incompleteness Claim faces serious difficulties, which have led at least one prominent Structuralist to back away from it. I have also argued that any attempt to salvage the first strain or make sense of the second results in claims that are very similar to the Dependence Claim. These findings give Structuralists and non-Structuralists alike a good reason to take the Dependence Claim seriously.

[15]See (Weatherson, 2006).

[16]Thanks to Ross Cameron and Robbie Williams for questions that forced me to clarify this.

3 The Dependence Claim

Almost all Structuralists commit themselves to the Dependence Claim, either explicitly or implicitly. We have already seen that the claim plays an important role in the characterization of Structuralism offered in (Shapiro, 2000). But the claim figures prominently in his more scholarly (Shapiro, 1997) as well. Here is one example.

> The structure is prior to the mathematical objects it contains, just as any organization is prior to the offices that constitute it. ((Shapiro, 1997), p. 78)

More generally, (Shapiro, 1997) contrasts structuralism with traditional platonism, which (he says) assimilates mathematical objects to ordinary physical objects. One fundamental feature of ordinary physical objects is that they are supposed to be ontologically independent of each other. For instance, my computer is supposed to be capable of existing even if my desk did not, and vice versa. According to Shapiro, the platonist claims that the same holds of mathematical objects; for instance, that "each natural number is independent of every other natural number" ((Shapiro, 1997), p. 72). Structuralism can then be distinguished from traditional platonism in that it denies that mathematical objects from the same structure are ontologically independent of each other.

But as Shapiro notes, it is difficult to spell out the relevant notion of independence in a satisfactory way. He takes some initial steps towards an explication, relying on the notion of *essence*. In particular, Shapiro suggests that the platonist view "may be that one can state the essence of each number without referring to the other numbers" (*ibid.*). He then comments that

> If this notion of independence can be made out, we structuralists would reject it. The essence of a natural number is its relations to other natural numbers. (*ibid.*)

Similar claims about the natural numbers being essentially related to each other are found other places, for instance

> [T]he essence of a natural number is the relations it has with other natural numbers. ((Shapiro, 1997), p. 5)

Now, the essence of an object is normally understood as what give the object its identity, what makes it the object that it is. Shapiro thus appears to be saying that the identity of a mathematical object depends on or is derived from the relations that it bears to other mathematical objects from the same structure. This means that a mathematical object has its identity only in virtue of occupying a certain position in the structures to which it belongs.

In this sense the object presupposes or depends on the structure to which it belongs. By contrast, ordinary concrete objects do not in this way presuppose or ontologically depend on the various structures that they enter into. For instance, although an apple occupies positions in various geometrical, biological, and economic structures, it is generally assumed that the apple has its identity independently of all these facts, such that it could occupy other positions in such structures, or perhaps not even enter into the structure at all, and still remain the object that it is.

Michael Resnik too commits himself to the Dependence Claim, although somewhat less explicitly than Shapiro does. For instance, in the following passage we find the same idea as in Shapiro, namely that mathematical objects owe their identities to their structural relationships to other mathematical objects.

> Mathematical objects [...] have their identities determined by their relationships to other positions in the structure to which they belong. ((Resnik, 1982), p. 95)

Let's now try to be more precise about what the Dependence Claim says. Consider the domain D of some mathematical structure. One aspect of the Dependence Claim is that in this structure *Objects Depend on Objects*:

(ODO) Each object in D depends on every other object in D.

For instance, each natural number is said to depend on all the other natural numbers in a way that it does not depend on, say, sets. It follows that it is impossible for one natural number to exist without all the others existing as well. Another aspect of the Dependence Claim is that in mathematical structures *Objects Depend on Structures*:

(ODS) Each mathematical object depends on the structure to which it belongs.

ODS has much the same effect as ODO. In particular, ODS ensures that it is impossible for one object from a particular mathematical structure to exist without all other objects from this structure existing as well. For if one object from some structure exists, then by ODS so too does its structure. But since a structure involves all of its positions, a structure cannot exist without all of its positions existing as well. Putting these claims together it follows that, if one object from a mathematical structure exists, then so do all the others.

If defensible, ODO would be a very substantial metaphysical discovery. It would mean that mathematical objects are subject to an *upward dependence* in that they depend on the structures to which they belong. If correct, this may point to a fundamental difference between the mathematical realm and the physical, where an opposite form of *downward dependence* appears to dominate. For most structures composed of physical objects appear to depend on their constituents, rather than the other way round.[17]

Both of the claims ODO and ODS are endorsed by Stewart Shapiro, who uses them as a cornerstone of his characterization of Structuralism. We saw one clear example of this in the following passage, already quoted in Section 1:

> The number 2 is no more and no less than the second position in the natural number structure; and 6 is the sixth position. Neither of them has any independence from the structure in which they are positions, and as positions in this structure, neither number is independent of the other. ((Shapiro, 2000), p. 258)

4 Recent objections to the Dependence Claim

Several philosophers have recently objected to the Dependence Claim. Before discussing these objections it will be instructive to consider a related but somewhat simpler objection, which as far as I know hasn't been developed in print. This objection is based on the claim that there cannot be circular relations of dependency. The objector can attempt to motivate this claim by observing that when an object b depends on another object a, then a must be "prior" to b. But then a cannot in turn depend on b, since two objects cannot be "prior" to each other in one and the same sense. The objector thus arrives at the claim that there cannot be any cyclical relations of dependency. The objector may also attempt to motivate the stronger claim that the notion of dependence must be well-founded:[18]

(WF) The dependence relation must be well-founded.

Armed with either the non-circularity requirement or the stronger well-foundedness requirement, the objector can reject the dependence claim ODO as unacceptable. For this claim says that mathematical objects stand in circular relations of dependency.

However, even if this objection could be made out—and that is a very big 'if'—it is doubtful that it would do much damage to Structuralism. For even if forced to give up ODO, Structuralists would still have the closely related dependence claim ODS to fall back on. To see this, recall that ODS postulates only a one-way dependence of objects upon the structure

[17]There may well be exceptions. For instance, a quantum particle in an entangled state is arguably ontologically dependent on the entire entangled system.

[18](Lowe, 2003) and (Lowe, 2005) defend the ban on cyclical relations of dependency. (Fine, 1995) defends a qualified form of the principle (WF). This principle is also implicit in (Hellman, 2001) and (MacBride, 2006), to be discussed below.

to which they belong. There is thus no obvious conflict between ODS and the requirement that the dependence relation be non-circular or even well-founded. Moreover, even on its own ODS gives Structuralists almost everything that they want. For we saw that ODS is just as effective as ODO in ensuring that, if one object from a mathematical structure exists, then so do all other objects from this stucture. Furthermore, because ODS is a claim about upward dependence, it allows Structuralists to sharply distinguish mathematical objects from ordinary concrete objects, which are subject to a downward dependence.

Geoffrey Hellman and Fraser MacBride have developed more sophisticated objections intended to undermine not just ODO but also ODS. According to these objections, even the upward dependence of ODS would introduce an impermissible circular dependency. Like the simple objection just discussed, Hellman and MacBride's objections begin by assuming that the dependence relation must be well-founded or at least non-circular. They then continue by defending (two different versions of) a completely general metaphysical thesis of downward dependence, roughly to the effect that any structure whatsoever depends on the objects that it involves. If (either version of) this general thesis is correct, then this will undermine ODS as well. But I will now go on to argue that both versions of the thesis of universal downward dependence are highly problematic.

Hellman's argument focuses on one particular example, namely the abstract structure shared by all systems that satisfy the Peano-Dedekind axioms. According to Structuralists, the positions in this abstract structure

> are entirely determined by the successor function [...], and derivative from it in the sense of being identified merely as the terms of the ordering induced by [this successor function]. ((Hellman, 2001), pp. 193-194)

Provided that the crucial words 'determined' and 'identified' are understood in a metaphysical rather than an epistemological sense, this is a fair characterization of the Structuralist view. But Hellman has misgivings about this view, as brought out in the following question.

> But if the relata are not already given but depend for their very identity upon a given ordering, what content is there to talk of 'the ordering'? (*ibid.*)

After expressing doubts about Structuralists' ability to answer this question, Hellman concludes as follows.[19]

> This, I submit, is a vicious circularity: in a nutshell, to understand the relata, we

[19] Another similar passage is found in (Hellman, 2005), p. 545.

> must be given the relation, but to understand the relation, we must already have access to the relata. (*ibid.*)

This conclusion is confusing, given the unexplained shift from metaphysical to epistemological vocabulary. Since the Structuralists clearly intend the Dependence Claim to be a metaphysical claim, the most charitable reading of the passage will presumably be to translate it back into purely and unambiguously metaphysical vocabulary. I thus propose that Hellman's claims about what is required in order to understand an entity be interpreted as claims about how the identity of this entity is *grounded*. When the above passage is translated in this way, it yields something like the following.

> This, I submit, is a vicious circularity: the Structuralists claim that the identity of the relata is grounded in that of the relation; but any grounding of the identity of a relation presupposes that the relata have already and independently had their identities grounded.

If this interpretation is on the right track, then Hellman is relying on the following premise, according to which relations in a certain sense depend upon their relata:

(RDO_1) The identity of any relation on a domain D presupposes that the individual objects from D have already and independently had their identities grounded.

This premise is potentially very damaging to Structuralism. For recall that a structure is a system of abstract objects that stand in purely structural relationships to each other. So if the relations involved in the structure depend on their relata in the way asserted by RDO_1, then so will the structure itself. Given the requirement that the dependence relation be well-founded, this downward dependence would thus contradict the upward dependence of ODS.

However, no defense of the thesis RDO_1 is provided. Perhaps the thesis is meant to be intuitively obvious. If so, Structuralists can respond that this intuition is just wrong and that mathematics provides counterexamples. Or perhaps the thesis is meant to be supported by examples. For instance, the identity of the relation of *being cousins* seems to presuppose that the relata—human beings—have already and independently had their identities grounded. For in order to say anything substantive about what relation this is, we will have to specify how two human beings—already individuated—have to be related in order for the relation to apply. However, this defense of RDO_1 would be plainly question begging against the Structuralists, whose point is precisely that mathematical objects differ from ordinary physical objects by being subject to an upward rather than a downward dependence. I conclude that no satisfactory defense of RDO_1 has been offered.

MacBride's argument—which he develops without fully endorsing[20]—differs somewhat from Hellman's. It is nicely summarized in the following passage:

> In order for objects to be eligible to serve as the terms of a [...] relation they must be independently constituted as numerically diverse. Speaking figuratively, they must be numerically diverse 'before' the relation can obtain; if they are not constituted independently of the obtaining of a [...] relation then there are simply no items available for the relation in question to obtain between. ((MacBride, 2006), p. 67)

The crucial premise employed here is a second form of universal downward dependence:

(RDO_2) The obtaining of any relation presupposes that the objects it relates have already had their identities grounded.

Much the same can be said about this premise as about RDO_1: Although it would clinch the anti-Structuralists' argument, no defense of the premise is provided, and any appeal to intuitive obviousness or to examples would have little suasive power vis-à-vis Structuralists.

5 A counterexample to the Dependence Claim

We have seen that two extreme views tend to dominate the debate: the Structuralists' claim that *all* mathematical objects depend on the structures to which they belong, and the anti-Structuralists' claim that *no* mathematical objects are subject to any such upward dependency. I will now provide counterexamples to each of these extreme views. My own position will thus be a compromise according to which some mathematical objects depend on their structures, whereas others don't.

This sort of compromise view is not new. Already Kant distinguished between two sorts of totalities with different dependence properties.[21] A *totum syntheticum* is a totality "synthesized" from pre-existing parts, which therefore depends on these parts. A *totum analyticum*, on the other hand, is a totality which is prior to its constituents and whose constituents therefore depend on the whole. Kant's principal examples of this sort of totality were space and time. For instance, he argued that the totality of space is prior to all of its different

[20] Indeed, (MacBride, 2005), p. 581 expresses serious reservations about this kind of argument.

[21] See (Allison, 1986), p. 43, who in turn cites Kant's *Reflexion* 393.

sub-spaces. A more recent example is found in the work of Charles Parsons. After articulating and defending a Structuralist view of "pure" mathematical objects, (Parsons, 1990) argues that Structuralism doesn't give a correct description of more basic "quasi-concrete" mathematical objects, such as geometrical figures and linguistic types, which have canonical representations in the concrete.[22] In order to refute the two extreme views, it would therefore suffice to defend either of these two earlier compromise views. However, I will be concerned with different examples and will end up demarcating the scope of Structuralism in a new and different way.

I believe sets provide an example of mathematical objects that aren't subject to the upward dependence of the theses ODO and ODS. According to the prevailing iterative conception, sets are "formed from" their elements. The relation between a set and its elements is thus asymmetric, because the elements must be "available" before the set can be formed, whereas the set need not be—and indeed cannot be—"available" before its elements are formed. A set thus appears to depend on its elements in a way that the elements don't depend on the set.[23] This asymmetry is perhaps most striking in the case of singletons. The identity of a singleton depends on that of its single element. For in order to specify what singleton we are talking about, we need to specify what its single element is. But the identity of the singleton does not depend on any other objects or on the hierarchy of sets. For an exhaustive account can be given of the identity of the singleton without mentioning any objects other than its single element.

This asymmetric dependence is in fact a very good thing, as there are all kinds of difficult questions about the higher reaches of the hierarchy of sets. How far does the hierarchy extend? Are the different stages rich enough for the Continuum Hypothesis to fail? It would be a pity if very simple sets, such as the empty set and its singleton, depended on the entire hierarchy of sets and their identities could therefore not be completely known before these hard questions had been answered. But fortunately the situation is the reverse. In particular, we can give an exhaustive account of the identity of the empty set and its singleton without even mentioning infinite sets.[24]

How can the Structuralist respond to this argument? One response would be to argue that we should abandon the iterative conception of set and adopt some alternative structuralist

[22]See (Parsons, 1990), pp. 337-338.

[23]See (Boolos, 1971) for a classic statement of the iterative conception of set and the idea that the elements of a set must be "available" before a set can be formed. The asymmetric dependence of a set on its elements is defended in (Fine, 1995) and (Lowe, 2005).

[24]I suspect it is possible to defend an analogous view of the natural numbers, according to which one natural number depends on its predecessors but not vice versa.

conception according to which all sets are individuated simultaneously. Although I am inclined to concede that such an alternative Structuralist conception of set is possible, I don't think this response has much force. For my present claim is simply that the ordinary iterative conception of set provides us with a counterexample to the dependence theses ODO and ODS. And this claim cannot be undermined by the development of any *alternative* conception of set.

A second and more promising response would be to challenge my claim that the ordinary iterative conception of set is non-Structuralist. This strategy is pursued in (Parsons, 1994) (which as far as I know is the only response to the present problem to be found in the literature). Parsons first argues that ZFC set theory rests on several distinct intuitions and cannot be reduced to any one of these. First there are combinatorial intuitions, associated with the conception of sets as collections. These intuitions support an ontologically rich membership relation which sustains an asymmetric dependence of a set upon its elements. Next there is the idea of limitation of size, which plays an important role in the justification of the Axiom of Replacement. Parsons then argues that, since no one unified set of intuitions can be used to motivate all of ZFC, a Structuralist conception of sets is more plausible. However, even if Parsons was right that ZFC cannot be motivated by one unified set of intuitions, this would not undermine my present argument. For all I need is that one strand of the iterative conception provides a non-Structuralist conception of sets, not that this strand suffices to motivate *all* the axioms of ZFC. And it seems clear that the combinatorial intuitions suffice to motivate at least the theory of hereditarily finite sets, which I can then use as my counterexample.

A third response would be to scrutinize the notion of dependence rather than the iterative conception of set. Perhaps a better understanding of the notion of dependence will show that sets don't constitute a counterexample after all? I agree that the intuitive notion of dependence that we have relied on needs to be analyzed. I therefore propose an analysis in Section 7 and show that on this analysis, sets do indeed constitute a counterexample to the dependence claims ODO and ODS. Structuralists are obviously free to develop their own alternative analysis of the notion of dependence. Prior to actually being presented with such an analysis, all I can do is express my grave doubts that any such analysis can manage to overturn our strong conviction that sets depend asymmetrically on their elements.

6 Examples where the Dependence Claim holds

I will now describe some examples of mathematical objects of which the Dependence Claim is true. These will also serve as counterexamples to the anti-Structuralists' downward dependence theses RDO_1 and RDO_2. I begin with a loose and intuitive treatment of a simple example, before turning to a more systematic exposition. Consider the unique group with two elements. We may denote these elements by α and β and give an exhaustive characterization of the group by means of the following multiplication table:

$*$	α	β
α	α	β
β	β	α

Provided we are willing to accept α and β as bona fide mathematical objects, this provides an example where the Dependence Claim holds. ODO holds because α and β are individuated simultaneously and relative to each other. Each object is what it is only by entering into certain relationships with the objects that make up the group. ODS holds for much the same reason: Since α and β are nothing but positions in this structure, their identity cannot be characterized without also characterizing the identity of the entire structure.

Let's now do things a bit more carefully and generally. (Readers who are less technically inclined may want to skim until the last two paragraphs of this section.) Let R and R' be n-place relations. Say that R and R' are *isomorphic* iff there is a one-to-one mapping f from the field of R onto the field of R' such that $Rx_1 \ldots x_n$ iff $R'f(x_1) \ldots f(x_n)$. We symbolize this as $R \cong R'$. We can now define *isomorphism types* of relations by the following abstraction principle:

$$\bar{R} = \bar{R}' \leftrightarrow R \cong R' \qquad (1)$$

But we have to be careful here; for without any restrictions, (1) leads straight to paradox.[25] Fortunately, a variety of restrictions are known that would shield us from paradox. The simplest restriction is just to let the relation-variables range only over sets and to let their isomorphism types be non-sets. Although this is not entirely satisfactory in general, it suffices for my present goal of providing examples of one-way upward dependence.[26]

I next claim that we can use the isomorphism types of relations to represent what I in Section 3 called *Dedekind abstraction*, that is, the operation that maps a system to its abstract structure. Recall also that Shapiro characterizes the structure of a system based on a domain D and relations $R_1, \ldots, R_n$ as "the abstract form" of this system. Let R be the product relation $R_1 \times \ldots \times R_n$ (where the product $S \times T$ of an m-place relation S and an n-place relation T is the $m+n$-place relation that holds of $x_1, \ldots, x_{m+n}$ just in case S

[25]It will allow us to derive Burali-Forti's paradox; see (Hazen, 1985), pp. 253-254.

[26]I attempt to develop a more systematic and satisfactory solution in (Linnebo,).

holds of $x_1, \ldots x_m$ and T holds of $x_{m+1}, \ldots x_{m+n}$). The single relation R can be regarded as a complete representation of the entire system with which we began, including all of the relations $R_1, \ldots, R_n$. To see why, consider the example of an algebraic ring. (A ring is a system consisting of a domain D on which are defined three-place relations ADD and MULT such that the usual axioms for addition and multiplication are satisfied. Paradigm examples are the familiar number systems $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$.) But the system associated with a ring is completely represented by means of an "addition table" and a "multiplication table" defined on the relevant domain. The product relation R is just a way of combining these two tables into one.

Given this way of representing particular systems, we can represent their structure or abstract form simply as $\bar{R}$. Where the particular system R has offices that are filled by independently individuated occupants, $\bar{R}$ is left with nothing but the offices themselves: everything else has been abstracted away from. Note that our construction of a "product system" is needed in order to keep track of how the same offices figure in the isomorphism types of the different relations R_i even when most other distinguishing features of this office has been abstracted away from.

A structure S is said to be *rigid* when the only isomorphism of S onto itself is that given by the identity mapping. When the structure of R is rigid, its various offices can be regarded as objects in their own right. Let $\tau(x, R)$ be the abstract office occupied by x in R. Such abstract offices can then be individuated in the following way:

(2) $$\tau(x, R) = \tau(x', R') \leftrightarrow \exists f[f : R \cong R' \wedge f(x) = x']$$

For any realization R of a rigid abstract structure S, there is a unique isomorphism between the occupants of the offices of R and the abstract offices of the structure S.[27] This allows us to define a natural *predicational interpretation* of an abstract structure S. We can define S to hold of the abstract offices $\tau(x_1, R_1), \ldots, \tau(x_n, R_n)$ iff there is a realization R of the abstract structure S which holds of $u_1, \ldots, u_n$, where for each i the office occupied by u_i in R corresponds to that occupied by x_i in R_i. This definition is permissible because it does not depend on the choice of realizations R and $R_1, \ldots, R_n$ of the abstract structure S.

The Dependence Claim appears to be true of structures obtained by this form of Dedekind abstraction. In particular, the abstract offices appear to depend on the structure to which they belong: each such office has its identity solely in virtue of belonging to this particular

[27] What about the positions of non-rigid abstract structures? Are they too objects? If so, how are they individuated? For the purposes of establishing my compromise view, I need not take a stand on these difficult questions. For discussion, see (Keränen, 2001) and (Shapiro, 2006b).

structure. (This argument will be reinforced in the next section when the notion of dependence is analyzed.)

As far as I can see, the only interesting anti-Structuralist response to this argument would be to deny that there really are mathematical objects of the sort described in this section. It would take me too far afield to defend my argument against this challenge. I just note that the Structuralists have developed a variety of arguments why such mathematical objects should be accepted. Particularly powerful are the arguments to the effect that ordinary mathematical practice is committed to such objects.[28]

To sum up, the picture that emerges from this section and the previous one is that the Dependence Claim is true of some mathematical objects and false of others. This raises the question where to draw the line. Say that a structure is *algebraic* if it arises from the process of Dedekind-abstraction; that is, if it is just the abstract form of some system of objects and relations. I have argued that the Structuralists are partially right because the Dependence Claim gives a correct description of algebraic structures. And although such structures don't exhaust mathematics, they include the vast majority of structures studied in contemporary mathematics. This gives Structuralism a central place in the interpretation of mathematics. However, I have also argued that the Structuralists are wrong about (iterative) sets and that the scope of the Dependence Claim will have to be restricted accordingly.[29]

7 Analyzing the notion of dependence

The notion of dependence that we have relied on thus far is problematic. The most common analysis of the notion of dependence is probably the straightforward modal one, according to which x depends on y just in case it is impossible for x to exist without y existing as well. However, since pure mathematical objects are generally assumed to exist necessarily, this modal analysis is useless for our purposes. For instance, on this analysis the existence of 2 no more depends on that of 5 than on that of the empty set. So any notion of dependence suited to figure in the debate about Structuralism will have to be more fine-grained. But unfortunately, neither party to the debate about the Dependence Claim has been particularly forthcoming about how they understand the crucial notion of dependence.

The only feature of the notion that emerges as reasonably clear is that the dependence in question is supposed to be a matter of how one object depends on others *for its identity*.[30] And

[28] See e.g. (Parsons, 1990); (Shapiro, 1997), ch. 3; and (Resnik, 1997), ch. 10.

[29] I am also sympathetic to Parsons' claim that Structuralism is false of "quasi-concrete" mathematical objects, such as geometrical figures and linguistic types. See also footnote 24.

[30] See Sections 3 and 4.

this kind of dependence has been analyzed by metaphysicians. Two of the most sophisticated analyses are due to Kit Fine and E.J. Lowe.[31]

To explain Fine's analysis of the relation of dependency we first need to review his analysis of the notion of an essential property. A property F is *essential* to an object x just in case x could not have been the object it is without possessing the property F. For instance, it is essential to Socrates that he is human, and it is essential to the natural number 3 that it is abstract. The notion of an essential property cannot be reduced to modal notions, say by defining a property F to be essential to an object x just in case it is necessary that Fx (if x exists at all).[32] To see this, consider the relation between Socrates and his singleton. This relation holds in all possible worlds in which Socrates exists. But although it is essential to the singleton that it contains Socrates as an element, it is not essential to Socrates that he is an element of this singleton. Equipped with this analysis of essential properties, Fine says that an object x *ontologically depends* on another object y just in case y is a constituent of some essential property of x. For instance, the singleton of Socrates depends on Socrates because the property of having Socrates as an element is essential to it. But there is no dependency in the reverse direction because the property of being an element of a certain singleton is not essential to Socrates.

According to Lowe, x *depends for its identity* on y just in case there is a function f such that it is essential to x that $x = f(y)$. This yields the same verdict as Fine's on the above example. Since it is essential to the singleton of Socrates that it is the value of the singleton function applied to Socrates as argument, this singleton depends on Socrates. But since it is not essential to Socrates that he is the value of the sole-element-of function applied to the singleton as argument, there is no dependency in the reverse direction.

In fact, these two analyses draw on a shared underlying idea, namely that an object depends for its identity on another object just in case any individuation of the former object must proceed via the latter. (By "individuation" is meant here an explanation of the identity of an entity.[33]) Applied to sets this yields the familiar conclusion. In order to individuate a set we will have to specify its elements. It is therefore impossible to individuate the singleton of Socrates without proceeding via Socrates himself. But if a person allows of any informative individuation at all, this individuation certainly need not proceed via any set.

Recall that my present goal is to make plausible that there are natural and important dependency relations on which the claims made in Sections 5 and 6 are sustained. In pursuit

[31]See (Fine, 1995) and (Lowe, 2005).

[32]See (Fine, 1994).

[33]See (Lowe, 2003) for discussion.

of this goal I propose to make use of the following two definitions. Say that x *strongly depends* on y just in case any individuation of x must proceed via y. This is just the shared underlying idea mentioned above. Say that x *weakly depends* on y just in case any individuation of x must make use of entities which also suffice to individuate y. For instance, a set x weakly depends on each of its subsets. For any individuation of x must make use of x's elements. But these elements suffice to individuate any subset of x as well. This weak dependency relation has (as far as I know) received little or no attention, despite being closely connected with Fine's and Lowe's shared underlying idea. Although both notions need further explication in order to be fully satisfactory, they suffice for present purposes.

We can now observe that the claims made in Section 5 are sustained on both the strong and the weak notion of dependency. We have already seen that a set is strongly dependent on its elements but that the elements are not strongly dependent on the set. But in fact the elements of a set are not even weakly dependent on the set. For in order to individuate a set, we need to draw on additional entities not needed to individuate its elements, namely the elements themselves. Sets thus provide a counterexample to ODO for either notion of dependence. Sets also provide a counterexample to ODS for either notion of dependence. No set is strongly dependent on the structure of the entire universe of sets. For every set can be individuated without proceeding via this structure. Nor is any set weakly dependent on this structure. For the entities needed to individuate a set fall (quite literally) infinitely far short of those needed to individuate this structure.

In Section 6 I claimed that the offices of an abstract structure S depend on each other and on the structure S. We can now observe that these claims are sustained on the weak notion of dependency. To see this, consider the abstract office individuated via an ordered pair $\langle x, R\rangle$, where R is some particular system that realizes the abstract structure S. As part of the realization R we are also given occupants of all the other R-offices. We thus have available the entities needed to individuate any of the other abstract offices. This means that each office weakly depends on all the others. We can also show that every office of an abstract structure weakly depends on the structure itself. For in order to individuate such an office we need a realization of the structure. But this is also all we need to individuate the relevant abstract structure itself. (For completeness, we may also observe that the converse holds. For the realization needed to individuate the abstract structure contains all the entities needed to individuate its abstract offices as well.)

I conclude that my counterexamples to the two extreme views on the dependence relations of mathematical objects are borne out on a plausible analysis of the notion of dependence.[34]

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