**Review of Stewart Shapiro, *Philosophy of Mathematics: Structure and Ontology***

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This book is an important contribution to the philosophy of mathematics. It aims to clarify and answer questions about realism in connection with mathematics, in particular whether there exist mathematical objects (ontological realism) and whether all meaningful mathematical statements have objective and determinate truth-values (truth-value realism). The author develops a novel *structuralist* account of mathematics that answers both questions affirmatively. By regarding mathematics as 'the science of structure' (p. 5), he attempts to render both forms of realism naturalistically respectable. The resulting philosophy of mathematics is extremely interesting and deserves the attention of anyone with a serious interest in the field.

The book is not an easy read, however. The prose is often quite dense, and although this is in part due to the difficulty of subject and the sheer number of ideas discussed, part of the blame must also be attributed to the fact that the discussion occasionally is poorly organized and that it can be hard to tell exactly what the author's view is. The introductory chapter is particularly dense and would serve better as a summary of the book than an introduction to it. Despite these imperfections, however, a careful study of the book will be enormously rewarding to anyone with some prior exposure to the field.

In what follows, I will go through the book, focusing on what I take to be its main themes.

## 1. Realism and anti-realism

Shapiro is dissatisfied with the traditional characterizations of ontological realism and truth-value realism alluded to above. In the first two chapters he attempts to give content to various realist claims by reconstruing them as 'competing, sober *programs*' (p. 6).

Chapter 1 begins with some examples of philosophical debates that have raged in and around mathematics. Should the law of excluded middle be accepted in all contexts? Are impredicative

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definitions acceptable in mathematics? Is the axiom of choice acceptable? Should functions be understood extensionally, as arbitrary mappings of one set into another, or in a more algorithmetic fashion? Shapiro describes how both sides of these debates made extensive use of philosophical arguments. The realist view that mathematical objects have objective existence was used to support the contested principles, and opposing anti-realist views, to support their negations. Let's therefore call the contested principles *the realist principles*. Today, the realist principles are firmly entrenched in mathematical practice. But Shapiro argues this is no thanks to the original philosophical arguments that were leveled in their favor, but rather, that the debates were settled on purely mathematical grounds or by appeal to the pragmatic consideration that the realist principles make for a better and more fruitful mathematics.

Shapiro warns against thinking the current consensus in favor of the realist principles represents a vindication of philosophical realism and points out that two further questions need to be considered before any such conclusion can be drawn. First, what *is* philosophical realism? And second, what is the philosophical significance of the sociological fact that mathematicians now accept the realist principles? That is, what is the relation between mathematical practice and philosophical theorizing?

Concerning the first question, Chapter 2 distinguishes two forms of realism. First, there is *working realism*, which is characterized by the following claim (pp. 38-9):

(WR) Mathematical practice should conform to the realist principles.

All working realism says is that mathematics should be practiced *as if* traditional realism were true; in particular, the view is meant to be compatible with forms of anti-realism that are non-revisionary vis-à-vis contemporary mathematical practice.

Second, there is *philosophical realism*, which is a more ambitious view. Where working realism is silent on most philosophical questions (p. 40), philosophical realism sets out 'to characterize [the] subject matter [of mathematics] (if it has one); to account for how mathematics is learned, communicated, and extended; and to delineate the place of mathematics in our overall intellectual lives, science in particular' (p. 44). The philosophical realist answer to all these questions is built around a core which says that the standard formalization of mathematics, already embraced by working realism, should be taken at face value:

(PR) The standard formalization of mathematics accurately represents the semantic form of mathematical language; and thus formalized, most claims of contemporary mathematics are true.

It follows from (PR) that mathematical variables should be understood as ranging over a domain of mathematical objects, and mathematical sentences, as having 'genuine, bivalent, nonvacuous truth conditions' (p. 44). So philosophical realism incorporates realism both in ontology and in truth-value.

Concerning the second question—about the relation between mathematical practice and philosophical theorizing—Shapiro describes two opposing orientations. According to 'philosophy-first', the principles of mathematics receive their authority, if any, from philosophy. First we need a philosophical account of what mathematics is about; only then can we determine what qualifies as correct mathematical reasoning. The other orientation, 'philosophy-last', holds that mathematics is an autonomous science that doesn't need to borrow its authority from other disciplines. On this view, philosophers have no right to legislate mathematical practice but must always accept mathematicians' own judgment.

Shapiro's great respect for mathematics brings him close to philosophy-last (p. 30). But he is careful to avoid the more extreme forms of this orientation, according to which the only legitimate task of philosophy is to describe actual mathematical practice. Shapiro insists that philosophy must also interpret and make sense of mathematical practice, and that this may give rise to criticism of said practice. However, he concedes to philosophy-last that this criticism would have to be *internal* to mathematical practice (p. 32), that it would have to take 'as "data" that most of contemporary mathematics is correct' (p. 4). This concession seems to me excessive. Since the interpretation of mathematical practice cannot be entirely internal to this practice, I don't see why the criticism to which it gives rise must be either.

Since philosophy-last urges us to respect mathematical practice, and since contemporary mathematical practice accepts the realist principles, working realism follows immediately. However, Shapiro denies that philosophy-last implies *philosophical* realism, because this form of realism aims to give an extra-mathematical account of mathematical practice. I am not entirely convinced. It seems to me a real danger that philosophy-last *will* imply philosophical realism and thus obliterate any distinction between Shapiro's two forms of realism. The problem is that mathematicians don't just claim that mathematical practice should conform to the realist principles but that these principles are *literally true*.[1] And in claiming this, they come very close to endorsing (PR).

Whether mathematicians should really be regarded as endorsing (PR) will depend on what is meant by 'accurately represents the semantic form of mathematical language' and 'true'. If the notions of semantic form and truth employed in (PR) are understood in a deflationary way, it's hard to see how (PR) can go beyond mathematicians' claim that the realist principles are literally true. Shapiro appreciates this and admits that one 'ought to feel cheated' if this is all philosophical realism comes to (p. 49). So it is incumbent upon the philosophical realist to spell out some richer, non-deflationary understanding of the notions of semantic form and truth which do not in this way trivialize philosophical realism. It is therefore with some surprise and disappointment I read Shapiro's (admirably honest) confession that he has 'nothing to offer on this question' (p. 48). This leaves his central distinction between working realism and philosophical realism purely programmatic and in danger of collapsing.

## 2. Mathematical structuralism

Having accepted working realism and recognized a strong presumption in favor of philosophical realism, Chapters 3 and 4 turn to some problems confronting philosophical realism. As is common these days, Shapiro focuses on two challenges made famous by Paul Benacerraf.[2] The first challenge, which is epistemological, demands a naturalistic account of our purported knowledge of abstract mathematical objects. The second, which is metaphysical, suggests that if there is no determinate answer whether the natural number 2 is identical with the Zermelo ordinal $\{\{\emptyset\}\}$ or with the von

[1] Just as physicists claim it is literally true that there exist electrons.
[2] See Benacerraf [1965] and [1973].

Neumann ordinal {∅, {∅}}, then there can be no such object as 2. Shapiro intends his mathematical structuralism to provide, or at least make room for, responses to both these challenges. In this section I discuss Shapiro's characterization of mathematical structuralism and the extent to which it answers the metaphysical challenge. In the final section, I discuss Shapiro's response to the epistemological challenge.

In philosophy and elsewhere, the word 'structuralism' is used in a variety of different ways. But in the philosophy of mathematics, structuralism is usually taken to be a metaphysical view of the nature of (some) mathematical objects, roughly to the effect that they have no properties other than those they have in virtue of occupying positions in the mathematical structures to which they belong. This kind of structuralism must be distinguished from certain semantic and epistemological kinds that are important elsewhere in analytic philosophy. An example of semantic structuralism is Quine's thesis of the inscrutability of reference, which says that, when a radical translator ascribes a theory to an agent, only the *structure* of this theory can be scientifically justified, because the objects of the theory always can be permuted without disturbing the translation's evidential support.[3] An example of epistemological structuralism is Putnam's model-theoretic argument, which aims to show that on a metaphysical realist view, we would have no 'epistemic access' to the correspondence between words and non-linguistic objects, other than to its structure which can be variously instantiated. Note that both these examples of structuralism apply to all objects whatsoever, whereas mathematical structuralism points to a feature that appears to be peculiar to (some) mathematical objects.

Shapiro is not as clear as he ought to be on the relation between his own mathematical structuralism and these alternative semantic and epistemic kinds. In fact, in a number of passages he places all these kinds of structuralism side by side and moves without notice from one to the other.[4] But given the *prima facie* difference between these kinds of structuralism, doing so would require further argument. In fact, I will argue below that Shapiro's version of mathematical structuralism points to features that are peculiar to mathematical objects. If I am right, mathematical structuralism will differ more profoundly from the alternative kinds than Shapiro admits.

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[3] For a succinct statement and defense of this view, see Quine [1992].

Shapiro's discussion of mathematical structuralism can be organized around three core intuitions. He opens with what can be called *the Dependence Intuition*. Ordinary physical objects are ontologically independent, not only of us, but of each other. My cup, for instance, could have existed even if my bottle did not. An analogous ontological independence appears to be found in pure mathematics. The existence of the natural number 2, for instance, appears not to involve that of the empty set, nor vice versa. The Dependence Intuition denies that mathematical objects from *the same structure* are ontologically independent of each other in this way. The existence of the natural number 2, for instance, depends upon that of the other natural numbers. It makes no sense to say that 2 could have existed even if 5 did not. Shapiro suggests the reason is that the natural number structure is prior to its individual elements, such that if one element exists, all do (p. 78).

As Shapiro admits, it is hard to give a satisfactory explication of the Dependence Intuition. Since pure mathematical objects exist necessarily, the usual modal explication of ontological dependence (in terms of its being possible for one object to exist without the other) gets no foothold. For on this explication, the existence of 2 no more depends on that of 5 than on that of the empty set. Shapiro attempts to bypass this problem by appealing to *essences*. He suggests that the 'essence of a natural number is its *relations* to other natural numbers' and thus that the essence of one natural number cannot be 'stated' without referring to that of other natural numbers (p. 72).

Although this is an interesting idea, it isn't adequately developed. The common understanding of essences isn't available because it analyzes this notion in modal terms. It would be more promising to take the notion of essence as primitive or at least as irreducible to modal concepts, and I suspect this is what Shapiro has in mind. But if so, he doesn't say how the notion of essence is to be understood. One option is to employ Kit Fine's work on primitive essences.[5] Another option is to attempt to connect the notion of essence with that of individuation and thus to transform the Dependence Intuition into the (plausible) claim that reference to a pure mathematical object is possible only in the context of the structure to which it belongs.

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[4] See pp. 65-7, 141-2, and especially 257-61.

[5] See e.g. Fine [1994].

The second structuralist intuition, which I will call *the Scarce Properties Intuition*, has probably been the primary motivation for the recent wave of interest in mathematical structuralism. According to this intuition, there 'is no more to the individual numers "in themselves" than the relations they bear to each other' (p. 73). The numbers have no 'internal composition' or extra-structural properties; rather, all the properties they have are those they have in virtue of occupying positions in the natural number structure. In particular, there 'is no answer' to the question whether 2 is identical to Julius Caesar or whether 1 is a (set-theoretic) element of 4 (p. 79). (As Shapiro makes clear, this isn't to say that it cannot often be useful and very natural to *stipulate* answers to some such questions. But he insists this will always be 'a matter of decision not discovery' (p. 258).)

A natural explication of the Scarce Properties Intuition is that the natural numbers have only arithmetical properties and that, for this reason, science should be regimented in a many-sorted language, where arithmetical expressions form a sort of their own. Metaphysically, this would correspond to the claim that the natural numbers form their own *category*. Shapiro seems quite sympathetic with this explication.[6] However, it faces two problems.

The first problem, which isn't discussed in the book,[7] is that the explication is false. For instance, even the structuralist admits that the natural numbers are abstract; but abstractness isn't an arithmetical property. Moreoever, from the claim that the numbers are abstract an answer to the Caesar problem follows by Leibniz's law: Because 2 is abstract and Caesar isn't, these objects cannot be identical.

I believe this problem can be dealt with by distinguishing between *primary* and *secondary* properties of an object. (No Lockean connotation intended!) We began with the idea that our talk about objects naturally falls into sorts. Let the *primary properties* of an object *x* be the properties associated with the sort S to which *x* belongs. But as the present problem illustrates, we sometimes compare and contrast different sorts, and this often leads us to regard objects from different categories as numerically distinct and to ascribe to them new properties that they inherit from the

[6] See pp. 79-81 and 168-9.
[7] It is, however, discussed in Shapiro [2003].

sorts to which they belong. For instance, we say that numbers are abstract and that physical objects are concrete. Call these the *secondary* properties of the object $x$. Our original explication of the Scarce Properties Intuition can now be refined as the claim that the only *primary* properties of natural numbers are arithmetical ones.

The second problem, which occasionally surfaces in the book,[8] is that the Scarce Properties Intuition isn't adequately captured either by the original explication or by the refinement just proposed. To see this, assume for the sake of the argument that physical objects form a sort. (If they don't, pick some subclass that does.) The relation between the category of natural numbers and that of physical objects will then be perfectly symmetrical. It will be no more surprising that the numbers have only arithmetical properties than that physical objects have only physical properties. Moreover, if there is no answer to the question whether 2 is identical to Julius Caesar, this will be no more the fault of 2 than of Caesar. Thus, our explications of the Scarce Properties intuition provide no more reason to be a structuralist about numbers than about physical objects. Shapiro occasionally seems happy with this conclusion (pp. 258-61). But in fact, I think his discussion contains the seeds of at least two arguments that the Scarce Properties Intuition doesn't fully carry over to physical objects.

One argument is that on Shapiro's version of structuralism there is a plethora of mathematical structures: not only natural numbers but integers, rationals, reals, complex numbers, quaternions, and so on through the vast zoology of non-algebraic structures that modern mathematics provides. Each of these structures is its own category. In contrast, there is no such proliferation of categories in the realm of the concrete. So there must be something special about pure mathematics that is responsible for this proliferation.

A second, complimentary, argument is contained in the third structuralist intuition Shapiro draws upon, namely that the (primary) properties of pure mathematical object are purely *formal*, unlike the *substantive* properties possessed by concrete objects. I will call this *the Formality Intuition*. This intuition is captured by Shapiro's claim that the subject matter of pure mathematics are *structures*, where a structure is said to be 'the abstract form of a system' of objects and relations

[8] See especially pp. 80-1 and 258-61.

on these objects (p. 74). A structure can thus be instantiated by a variety of systems of more substantive objects and relations; for instance, the natural number structure can be instantiated by the sequence of ordinary numerals and by the sequence of strokes: |, ||, |||, etc. Conversely, a structure can be arrived at by *abstraction* from a system of more substantive objects and relations.

Shapiro makes a very interesting suggestion about what it means for a property to be formal, as opposed to substantive. Recall Tarski's characterization of a logical notion as one whose extension remains unchanged under every permutation of the domain.[9] Drawing on this idea, Shapiro suggests that a property is formal just in case 'it can be completely defined in a higher-order language, using only terminology that denotes Tarski-logical notions and other objecs and relations of the system, with the other objects and relations completely defined at the same time' (p. 99).

In sum, Shapiro's discussion of mathematical structuralism contains a wealth of ideas. Although many of these ideas would require substantial further development to be fully convincing, they form the beginning of a powerful metaphysical picture of (pure) mathematical objects, according to which these objects are radically different from ordinary concrete objects—perhaps more so, in fact, than Shapiro himself is willing to grant.[10]

**3. Mathematical structures and the Equivalence Thesis**

Central to Shapiro's discussion of structuralism is a distinction between two perspectives we can take on a structure and its places. On the one hand, we can regard the places of the structure as *offices* and use structural vocabulary to make claims about their occupants. To use one of Shapiro's favorite examples, we can talk about the baseball defense structure in order to make claims about the individual players: 'the shortstop today was the second basedman yesterday' (p. 82). On the other hand, we can regard the places of the structure as *objects* in their own right. This is what we do when we say that the vice president is president of the Senate and that the chess bishop moves on a diagonal.

[9] See Tarski [1986].
[10] Cf. my discussion at the opening of this section.

This distinction suggests an eliminative project: Perhaps all talk about places-as-objects can be eliminated in favor of talk about the places-as-offices. The claim that the bishop moves on a diagonal, for instance, can arguably be paraphrased as the generalization that in every chess system, a piece that occupies the bishop role moves on a diagonal. If successful, this eliminative project would allow a 'structuralism without structures', avoiding ontological commitment to either structures or reified places in structures.

Shapiro compares the relation of a mathematical structure to a system of objects thus structured to the relation of a universal to a particular instantiating it (p. 84). In this way he relates questions about the ontological status of structures to the ancient debate about the status of universals. The two main alternatives are said to be *ante rem* structuralism, which holds that a structure exists regardless of whether it has any prior realizations,[11] and *in re* structuralism, which hold that a structure exists only in virtue of having such realizations. An advocate of *in re* structuralism may be attracted to the eliminative project just mentioned.

The main problem confronting this eliminative project is what we may call the 'non-vacuity problem': What guarantees that there will be realizations of the familiar mathematical structures and thus that the generalizations that replace ordinary mathematical claims won't all be vacuously true? Shapiro argues we cannot count on the physical universe's being large enough to realize the relevant structures and that doing so would be unacceptable anyway by making mathematics hostage to contingent physical fact.

He considers three alternative responses. The *ontological option* is to postulate enough abstract objects for all the relevant structures to be instantiated and to offer some non-structuralist acount of these abstract objects (p. 87). Typically, the universe **V** of sets is chosen to play this role, and the technical details are then well known. The *modal option* suggests that the *possible* existence of a realization of the structure suffices (p. 88). Shapiro argues that this option is most plausible when the modalities are taken to be *logical*. In the recent literature, this option is associated with Hellman [1989], where many of the technical details are worked out. Finally, the *ante rem* option holds that

[11] According to *ante rem* structuralism, an *ante rem* structure instantiates itself; see e.g. p. 100.

structures exist regardless of whether they have prior realizations. Shapiro offers a useful axiomatization of this option, modeled upon second-order ZFC set theory but that adds to this the 'main principle behind structuralism', which states that 'If Φ is a coherent formula in a second-order language, then there is a structure that satisfies Φ' (p. 95).

The main question for the two non-set-theoretic options is how their central notions—possibility and coherence—are to be understood. Shapiro argues that our only serious handle on these notions is afforded by analyzing them in terms of the familiar set-theoretic notion of satisfiability (possibly in a set-theory that includes reflection principles). Indeed, one of the main claims of the book is that the three options are in some sense equivalent. I will call this the *Equivalence Thesis*.[12] (Shapiro nevertheless expresses a preference for the *ante rem* option as 'the most perspicuous and least artificial of the three' (p. 90).)

What exactly does the Equivalence Thesis say? At the very least, the thesis says that three options are *logically* equivalent. Shapiro establishes this by showing that the three options are intertranslatable, or, more precisely, that they are 'definitionally equivalent' (pp. 224-5). His argument, located in Section 3.4 and Chapter 7, contains useful technical details.

However, the Equivalence Thesis is supposed to have philosophical as well as technical significance. Shapiro claims, firstly, that the three options 'all say the same thing, using different primitives', and secondly, that his translations preserve epistemological status (p. 97). Both these claims are problematic. Although Shapiro explicitly refrains from claiming that his translations preserve meaning (p. 224), he offers no alternative, positive account of the sense in which the three options 'say the same thing'. We are therefore left with little more than the guiding idea that the three options yield the same 'structure of structures' (p. 90).

Concerning the epistemological claim, Shapiro seems to think this follows immediately from the claim about logical intertranslatability.[13] However, it isn't generally true that intertranslatable

[12] Shapiro often appears to qualify this claim. On p. 90, for instance, the Equivalence Thesis isn't asserted outright but is merely said to follow from 'the thesis of structuralism'. Elsewhere, Shapiro inserts 'in a sense' or some similar expression in front of the Equivalence Thesis; see e.g. 96. However, because other statements of the thesis are unqualified, I will assume no qualifiation is needed.
[13] See e.g. pp. 96 and 225-6.

theories share epistemic status. For instance, arithmetic and the theory of syntax are intertranslatable but arguably differ in epistemic status, the latter being based on intuition, but the former, not. Moreover, a theory of geometry will have one epistemic status when understood as a pure mathematical theory and another when taken to be about physical space. So Shapiro's epistemological claim needs to be qualified. A promising qualification is that intertranslatability ensures sameness of epistemic status when the theories in question are *purely formal* in the sense discussed at the end of Section 2.

But even when thus qualified, why should we believe this epistemological claim? Shapiro's most promising defense of it is that our ordinary, informal understanding of the notions of possibility and coherence is inadequate for the relevant applications to mathematics, and that our grasp of these notions, as applied to mathematics, 'is mediated by mathematics, set theory in particular' (p. 238). He gives the following example. According to the modal structuralist, 'first-order set theory together with the assertion that there are exactly 124 infinite cardinals less than the continuum is possible' (p. 237). But it is hard to see what grounds the modal structuralist could have for this claim if he didn't believe in set-theoretic models.

Shapiro is surely right that this is hard. However, I think it remains possible for the nominalist to dig in his heels and insist that the platonist's explanatory burden is *even greater*. The crucial difference, according to this nominalist line, is that the platonist has an additional commitment to the existence of abstract objects, which the nominalist avoids.[14]

The last section of Chapter 7 can be read as an attempt to counter this danger. Here Shapiro proposes a novel account of a theory's commitments, which is supposed to replace the orthodox Quinean account of ontological commitment with an account that accommodates the possibility of tradeoffs between ontology and ideology. The idea is that, 'If two theories invoke equivalent structures, then they are equivalent on the ontology/ideology scale' (p. 239). Clearly, on this alternative account the three options have the same commitments.

[14] For an example of this line, see Hellman [2001].

Although I am sympathetic with this alternative account of a theory's commitments,[15] I don't see that Shapiro gives an adequate defense of it. It remains mysterious how a theory with a certain ontological commitment can be equivalent to one that lacks this commitment. This would mean that the notion of an object isn't absolute but relative to language and/or theory. And the same would go for reference. For instance, on the *ante rem* option, the numerals genuinely refer to natural numbers (construed as reified positions in the natural number structure), whereas on the modal option, 'strictly speaking, there is no reference at all' (p. 141). But when two theories differ ontologically and semantically in these ways, how can they then be equivalent or 'say the same thing'? To dispel the lingering sense of mystery, Shapiro makes some suggestive remarks about Putnam's notion of 'conceptual relativity'. But no systematic account is worked out.

## 4. Epistemology and reference

Shapiro regards the task of providing an epistemology for mathematical realism as both 'the central problem facing the program' (p. 45) and where the 'deepest problems' lie (p. 36). The problem, presented in Benacerraf's classic 'Mathematical Truth' (Benacerraf [1973]), is how we can have knowledge of objects that we don't causally interact with. Although Shapiro dismisses the original version of this challenge as based on an outdated casual theory of knowledge, he argues that a challenge remains of providing a 'naturalized epistemology' of mathematics, that is, of providing an epistemological account that involves 'ony natural processes amenable to ordinary scientific scrutiny' (p. 110).

This challenge is addressed in Chapter 4. Focusing on *ante rem* realism, Shapiro describes three ways in which mathematical structures can be apprehended and knowledge about them can be obtained.

The first way has to do with pattern recognition. First, we learn perceptually that individual objects and systems of objects display a variety of patterns. For instance, we learn to recognize the

[15] I make use of similar ideas in Linnebo [2003].

pattern shared by all triples (which we may call the *3 pattern*). Then, by an act of abstraction, we move from systems of objects patterned to the pattern itself, now regarded as an object in its own right. This in turn allows us to discern new patterns, such as that instantiated by the system consisting of the 1 pattern, the 2 pattern, the 3 pattern, etc. In this way we arrive at the natural number structure. Shapiro doesn't attempt to speculate about the psychological mechanisms underlying this kind of learning but insists that nothing 'philosophically occult' is involved (p. 113). But although this sounds plausible, we need to know more about the epistemology of the crucial step from the perspective of places-as-offices, which has no abstract commitments, to that of places-as-objects, which *is* thus committed.

The second way is based on the idea that mathematical objects can be introduced by abstraction on an equivalence relation over some prior class of entities. The idea appears to be much the same as that at the heart of the Wright-Hale neo-logicist project. What is new is Shapiro's 'starkly un-Fregean' view that there is no 'single, fixed universe, divided into objects a priori' but that, at least in mathematics, the notions of object and identity are always relative to a background theory or framework (p. 127). The so-called Julius Caear-problem is thus dealt with by denying that there is a fact of the matter about cross-category identity statements, and Russell's paradox, by denying that there is a universal class from which the Russell class can be defined by Comprehension.

The third way is based on implicit definition. Recall the main principle of *ante rem* structuralism, which states that whenever $\Phi$ is a coherent formula, there is a structure that satisfies $\Phi$. Shapiro now invokes an epistemological counterpart of this principle which says that, by laying down an implicit definition and convincing ourselves of its coherence, we successfully refer to the structure it defines. Thus, 'schematic knowledge about how language works leads to knowledge about structures' (p. 140).

Clearly, this tight link between grasp of language and knowledge of mathematical structures distinguishes *ante rem* structuralism from more traditional forms of ontological realism, which insist on a greater distance between the language of mathematics and its subject matter. In fact, if Shapiro is right that knowledge of mathematical language suffices for 'access to' mathematical objects, it seems these objects must somehow be language constituted, that their existence must somehow be ensured by certain (objective) features of our mathematical language. But if so, Shapiro's ontological realism will be much less robust than his rhetoric suggests.

Although I suspect Shapiro would not welcome this conclusion, it seems to me it may point the way to a transformation of the traditional epistemological problem posed by mathematical realism. Instead of being primarily a problem about access to abstract objects, it now becomes a problem about our grasp of mathematical concepts. The central notion is that of coherence. A theory is coherent if it is in principle possible to assign a unique truth-value to each of its sentences, observing the principles of logic. But in the vast majority of cases, we're not actually able to determine what truth-value is assigned to a sentence. What, then, does our understanding of the theory consist in? Perhaps this problem, not Benacerraf's, ought to be the main epistemological question in connection with mathematics.[16]

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