Sets, Properties, and Unrestricted Quantification

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1 Introduction

Call a quantifier *unrestricted* if it ranges over absolutely all things: not just over all physical things or all things relevant to some particular utterance or discourse but over absolutely everything there is. *Prima facie*, unrestricted quantification seems to be perfectly coherent. For such quantification appears to be involved in a variety of claims that all normal human beings are capable of understanding. For instance, some basic logical and mathematical truths appear to involve unrestricted quantification, such as the truth that absolutely everything is self-identical and the truth that the empty set has absolutely no members. Various metaphysical views too appear to involve unrestricted quantification, such as the physicalist view that absolutely everything is physical.

However, the set-theoretic and semantic paradoxes have been used to challenge the coherence of unrestricted quantification. It has been argued that, whenever we form a conception of a certain range of quantification, this conception can be used to define further objects not in this range, thus establishing that the quantification wasn't unrestricted after all.[1]

This paper has two main goals. My first goal is to point out some problems with the most promising defense of unrestricted quantification developed to date. My second goal is to develop a better defense. The most promising defense of unrestricted quantification developed to date makes use of a hierarchy of types (Section 3). I show that there are some important semantic insights that type-theorists cannot express in full generality (Section 4). I argue that this problem is analogous to those faced by philosophers who deny the coherence of unrestricted quantification. My alternative defense of unrestricted quantification is based on a sharp distinction between sets and properties (Section 5). Sets are combinatorial entities, individuated by reference to their elements. Properties are intensional entities, individuated

[1] See for instance Dummett, 1981, chapter 16; Parsons, 1974; Parsons, 1977; and Glanzberg, 2004.

by reference to their membership criteria. I propose that predicates be assigned properties as their semantic values rather than sets. This allows us to express the semantic insights that the type-theorists failed to express. Instead the paradoxes threaten to re-emerge. I deal with this by imposing a restriction on the property comprehension scheme—a restriction which I argue flows naturally from the nature of properties (Section 7) and which I prove to be consistent relative to set theory (Appendix), but which still yields enough properties to allow the desired kind of semantic theorizing (Sections 6 and 8).

2 The Semantic Argument

I begin by outlining what I take to be the strongest argument against the coherence of unrestricted quantification. Since this argument is based on a demand for semantic explicitness, I will refer to it as *the Semantic Argument*.[2]

Given any legitimate first-order language $\mathcal{L}$, it must be possible to develop the semantics of $\mathcal{L}$ in a systematic and scientifically respectable way, without imposing any arbitrary restrictions on the ranges of its quantifiers. I will refer to such a semantic theory as a *general semantics*. The requirement that any legitimate language should admit a general semantics is *prima facie* perfectly reasonable. Granted, it is perhaps conceivable that *Semantic Pessimism* should be true: that is, that there should be legitimate languages whose semantics for some reason cannot be made explicit. But Semantic Pessimism should be a last resort. In semantics, as well as in any other area of inquiry, a sound scientific attitude demands that we seek general and informative explanations until the impossibility thereof has been firmly established.

Assume for contradiction that it is possible to quantify over absolutely everything. In order to develop a general semantics for $\mathcal{L}$, we need to generalize over interpretations of $\mathcal{L}$'s primitive non-logical expressions. Let 'P' be a monadic predicate of $\mathcal{L}$. Let F be any contentful predicate of our meta-language, which for present purposes we may assume to be English. Then it must be possible to interpret 'P' as meaning F. Or, to state this claim more precisely, let an *interpretation* be an assignment of suitable semantic values to the primitive non-logical expressions of $\mathcal{L}$. Then there must exist an interpretation I_F such that

(1) $$\forall x\, (I_F \text{ is an interpretation under which `}P\text{' applies to } x \leftrightarrow Fx)$$

where the universal quantifier '$\forall x$' ranges over absolutely everything.

[2] My formulation of this argument draws heavily on Williamson, 2003.

Now come two more controversial steps, which I label for future reference.

Sem1 An interpretation is an object.

By this is meant simply that interpretations are the kinds of entities that first-order variables can range over. This seems reasonable, at least at the outset. Given **Sem1**, it also seems reasonable to go on and make the following step.

Sem2 We can define a contentful predicate 'R' as follows:

(2) $$\forall x\,(Rx \leftrightarrow x \text{ is not an interpretation under which '}P\text{' applies to } x)$$

Having made the steps **Sem1** and **Sem2**, the rest of the argument is uncontroversial. First we put 'R' for 'F' in (1) and apply the definition of 'R' from (2) to get

(3) $$\forall x\,(I_R \text{ is an interpretation under which '}P\text{' applies to } x \leftrightarrow$$
$$x \text{ is not an interpretation under which '}P\text{' applies to } x).$$

Since the quantifier '$\forall x$' in (3) is assumed to range over absolutely everything, we can instantiate it with respect to I_R to get

(4) $$I_R \text{ is an interpretation under which '}P\text{' applies to } I_R \leftrightarrow$$
$$I_R \text{ is not an interpretation under which '}P\text{' applies to } I_R.$$

Since this is a contradiction, we must reject the assumption that we can quantify over absolutely everything.

Suppose that the Semantic Argument works.[3] What would this establish? The answer seems simple enough: it would establish that we cannot quantify over absolutely everything. But let's attempt to be a bit more precise. When I is an interpretation of our language, let $\forall_I$ and $\exists_I$ be the resulting interpretations of the quantifiers. Let $I \subseteq J$ abbreviate $\forall_I x \exists_J y(x = y)$, that is, the claim that all the objects that exist according to I also exist according to J. Then if unrestricted quantification is possible, there must be an interpretation that is maximal in this ordering. But what the Semantic Argument would establish, if successful, is that there is no such maximal interpretation but on the contrary, that every interpretation has a proper

[3] My discussion in this paragraph and the next draws on Fine, 2006, Section 2.

extension

(5) $$\forall I\, \exists J\, (I \subset J)$$

where $I \subset J$ is defined as $I \subseteq J \wedge \neg(J \subseteq I)$.

However, things are not as simple as they seem. If the thesis that unrestricted quantification is impossible is true, then this thesis must apply to its own statement as well. This means that the quantifier '$\forall I$' cannot be unrestricted but must range over some limited domain D. But then all (5) says is that every interpretation *in the domain* D can be properly extended. But this is compatible with the existence of a maximal interpretation *outside of* D. Thus, if we limit ourselves to restricted quantification, we cannot even properly state the thesis that unrestricted quantification is impossible. In order to properly state the thesis, we appear to need precisely what the thesis disallows.

We thus appear to be in an unacceptable situation. On the one hand we have an argument that absolutely unrestricted quantification is incoherent. On the other hand we appear unable to properly state the conclusion of this argument. Something has got to give. In what follows I discuss two strategies for defending the coherence of unrestricted quantification, each based on rejecting one of the premises of the Semantic Argument.[4]

3 Type-theoretic defenses of unrestricted quantification

Can the coherence of unrestricted quantification be defended against the Semantic Argument? The most promising class of defenses developed to date attempt to use higher-order logic to undermine the argument's first premise, **Sem1**.[5] This premise, which says that interpretations are *objects*, records the fact that we used first-order variables to range over interpretations. So if we instead use second-order variables to talk about interpretations, this premise will no longer be true. It has therefore been suggested that an interpretation be represented by means of a second-order variable I with the convention that under I, an object-language predicate 'P' applies to an object x just in case $I\langle \text{‘}P\text{’}, x\rangle$.[6] The definition of the Russell predicate 'R' in

[4]Another strategy, which will not be discussed here, is to look for a way in which the conclusion of the Semantic Argument can be properly stated after all. For some attempts to carry out this strategy, see Fine, 2006, Glanzberg, 2004, and Glanzberg, 2006.

[5]This kind of defense of unrestricted quantification was given a clear and powerful statement in Boolos, 1985, has since been endorsed in Cartwright, 1994 and Lewis, 1991, and has recently been developed with quite a lot of technical and philosophical detail in Rayo and Uzquiano, 1999, Williamson, 2003, Rayo and Williamson, 2003, and Rayo, 2006.

[6]As shown in Rayo and Uzquiano, 1999, we can also define a satisfaction predicate which holds of an interpretation I and the Gödel number of a formula ϕ just in case I satisfies ϕ. But this will require not

Sem2 must then be rejected on the ground that it confuses first- and second-order variables. This blocks the rest of the Semantic Argument.

But it would be premature for a defender of unrestricted quantification to declare victory.[7] For a simple modification of the Semantic Argument shows that it isn't sufficient to admit second-order quantification but that quantification of arbitrarily high (finite) orders is needed. We see this as follows. The Semantic Argument challenges us to develop a general semantics for some first-order language $\mathcal{L}_1$, say that of ZFC set theory. The response just outlined develops a general semantics for $\mathcal{L}_1$ in a second-order language $\mathcal{L}_2$. However, the Semantic Argument was based on the requirement that it be possible to develop a general semantics for *any* legitimate language. Now clearly, if the above response to the Semantic Argument is to succeed, the language $\mathcal{L}_2$ must itself be legitimate. But then the Semantic Argument will require that a general semantics be developed for $\mathcal{L}_2$ as well.

In order to do this, we need to adopt a language that is more expressive than $\mathcal{L}_2$. For in order to develop a general semantics for $\mathcal{L}_2$, we would among other things have to give a theory of truth for the language $\mathcal{L}_2$ of second-order ZFC when its first-order quantifiers range over absolutely all sets. But by Tarski's theorem on the undefinability of truth, this cannot be done within $\mathcal{L}_2$ itself. The most natural thing to do at this point is to adopt an extended language $\mathcal{L}_3$ that includes *third*-order quantification as well. The argument just given can then be applied to $\mathcal{L}_3$. Advocates of the response based on second-order logic will in this way be forced up through the hierarchy of higher and higher levels of quantification.[8]

This response will thus need at least what is known as *simple type theory* (or *ST* for short). In the language $\mathcal{L}_{\mathrm{ST}}$ of this theory, every variables and every argument place of a predicate has a natural number as an upper index. These indices are called *types*. A formula is well-formed only if its types mesh, in the sense that only variables of type n occur at argument places of type n, and that only variables of type $n+1$ are predicated of variables of type n.[9] The theory ST contains the usual rules for the connectives and quantifiers, as well

only second-order quantification but *second-order predicates* (that is, predicates that apply to second-order variables in the same way as first-order predicates apply to first-order variables).

[7]This is openly acknowledged by at least some of the defenders of this response. See for instance the papers by Rayo, Uzquiano, and Williamson cited in footnote 5.

[8]Strictly speaking, the situation is a bit more complicated. A language with n-th order quantification may or may not contain predicates taking n-th order variables as arguments. If it does (doesn't), let's say that it is a *full* (*basic*) n-th order language. We can then prove that one cannot develop a general semantics for a basic (full) n-th order language in any basic (full) n-language, but that one can develop a general semantics for a basic n-th order language in a full n-th order language and for a full n-th order language in a basic $(n+1)$-th order language. For discussion, see Rayo, 2006. But my claim remains valid: The requirement that it be possible to develop a general semantics for any permissible language forces advocates of the second-order response up through the hierarchy of types.

[9]That is, whenever we have an expression of the form $\ulcorner \mathbf{v_1}(\mathbf{v_2}) \urcorner$, where $\mathbf{v_1}$ and $\mathbf{v_2}$ are variables, then the

as a full impredicative comprehension scheme for each type.

How should advocates of this response (or *type-theorists* as I will henceforth call them) interpret the formal theory ST? There are two main alternatives, one based on plural quantifiers, and another based on quantification into concept position. I will refer to the former as *pluralism* and the latter as *conceptualism*.

Pluralism uses as its point of departure George Boolos's interpretation of second-order quantifiers in terms of natural language plural quantifiers: '$\forall v^2 \ldots$' is rendered as 'whenever there are some things vv then ...', '$\exists v^2 \ldots$' as 'there are some things vv such that ...', and predication $v^2(v^1)$ as the claim that v is one of vv (in symbols $v \prec vv$). But what about quantifiers of orders higher than two? For instance, in order to extend Boolos's idea to third-order quantifiers we would need quantifiers that stand to ordinary plural quantifiers as ordinary plural quantifiers stand to singular quantifiers. Let's call such quantifiers *second-level* plural quantifiers. Do such quantifiers exist? The answer may be negative if by 'existence' we here mean *existence in natural language*. However, existence in natural language is at best a *sufficient* criterion for something to be a legitimate device in semantic theorizing; it is certainly not a *necessary* condition. This opens the possibility of independent arguments for the legitimacy of higher-level plural quantifiers.[10]

Conceptualism interprets the formal theory ST in a much more Fregean way: the second-order quantifiers range over ordinary concepts taking objects as their arguments, third-order quantifiers over concepts of such concepts, and so on. But when glossing this interpretation, the conceptualist has to be stricter than Frege himself was. For instance, the conceptualist cannot say that the second-order quantifiers range over concepts; for both argument places of the predicate 'ranges over' are first-order and hence apply only to *objects*. It therefore seems that the conceptualist interpretation of the language of higher-order logic can be adequately explained only using $\mathcal{L}_{\mathrm{ST}}$ *interpreted in precisely the way at issue.* One prominent conceptualist, Timothy Williamson, therefore suggests that "[w]e may have to learn [higher-order] languages by the direct method, not by translating them into a language with which we are already familiar."[11]

type of $\mathbf{v_1}$ is one higher than that of $\mathbf{v_2}$. (Throughout this paper, meta-linguistic variables will be indicated by means of boldface.)

[10]Such arguments are given in Hazen, 1997, Linnebo, 2004, and (at greater length) Rayo, 2006.

[11]See Williamson, 2003, p. 459, where this claim is made about *second-order* languages. But his "fifth point" on p. 457 makes it clear the same has to hold of higher-order languages more generally.

4 A problem with the type-theoretic defenses

I will now argue that all type-theorists face a serious problem: On their view, there are certain deep and interesting semantic insights that cannot properly be expressed.[12]

These insights all involve the notion of a *semantic value*, which plays a fundamental role in modern semantics and philosophy of language. Very briefly, this notion can be explained as follows. Each component of a sentence appears to make some definite contribution to the truth or falsity of the sentence. This contribution is its *semantic value*. It further appears that the truth or falsity of the sentence is determined as a function of the semantic values of its constituents. This is the *Principle of Compositionality*. In classical semantics, the semantic value of a sentence is taken to be its truth-value, and the semantic value of a proper name is taken to be its referent. Once we have fixed the kinds of semantic values assigned to sentences and proper names, it is easy to determine what kinds of semantic values to assign to expressions of other syntactic categories. For instance, the semantic value of a monadic first-order predicate will have to be a function from objects to truth-values.

We have seen that the type-theorists respond to the Semantic Argument by denying **Sem1**, which says that interpretations are objects. We have also seen that this forces the type-theorists up the hierarchy of higher and higher levels of quantification. This commits the type-theorists to a deep and interesting semantic view. On this view, proper names make a distinctive kind of semantic contribution to sentences in which they occur, namely the objects to which they refer. Likewise, monadic first-order predicates make a distinctive kind of semantic contribution: loosely speaking, a function from objects to truth-values, but, according to the type-theorists, properly represented only by means of second-order variables. And so it continues up through the types: for each natural number n, monadic n'th-order predicates make a distinctive kind of semantic contribution, properly represented only by means of $(n + 1)$'th-order variables. The type-theorist is therefore committed to generalizations of the following sorts.

- *Infinity.* There are infinitely many different kinds of semantic value.
- *Unique Existence.* Every expression of every syntactic category has a semantic value which is unique, not just within a particular type, but across all types.
- *Compositionality.* The semantic value of a complex expression is determined as a function of the semantic values of the expression's simpler constituents.

[12] Similar arguments are familiar from the literature. For a nice example, see Gödel, 1944, p. 466.

However, type-theorists are prevented from properly expressing any of these insights. For according to type-theory itself, no variable can range over more than one level of the type-theoretic hierarchy. But the above insights essentially involve generalizations across types. Type-theorists thus face expressive limitations embarrassingly similar to those that they set out to avoid in the first place.[13]

Can the type-theorists express the relevant insights in a more devious way? One suggestion is that, although these insights cannot be said straight out, they can nevertheless be *shown*, say by the logical forms of the things that can be properly said. (A similar view is often attributed to Wittgenstein's *Tractatus*.) Another suggestion is that the relevant insights can be expressed in what Carnap calls "the formal mode," that is, by talking about the type-theoretic distinctions only at the level of syntax, never at the level of semantic values. However, it seems doubtful that the former suggestion can be developed without resorting to unpalatable mysticism, and the latter, without denying the possibility of semantics in any sense worthy of the term. Moreover, both suggestions run completely counter to the spirit of the Semantic Argument. By giving up on semantic explicitness, these suggestions have much more in common with the view I called Semantic Pessimism.

A third and more promising suggestion is that the generalizations in question can be expressed by some sort of *schematic generality*. This was the idea behind Russell's notion of *typical ambiguity*. But this suggestion too is problematic. Firstly, a notion of schematic generality would have to be developed which doesn't collapse to ordinary universal quantification. Given that the two sorts of generality are subject to the same introduction and elimination rules, it is unclear whether this can be done. Secondly, since schematic generalities involve free variables but not quantifiers, they yield the expressive power of universal quantification but not of existential.[14] But each of the above three generalizations about the type-theorists' hierarchy make essential use of existential quantifiers in addition to universal ones. Moreover, this expressive limitation means that claims involving schematic generality cannot properly be negated or figure in the antecedents of conditionals. This sits very poorly with the spirit of semantic explicitness associated with the Semantic Argument.

It seems to me that the problem is best dealt with head on by avoiding all type-theoretic restrictions. The simplest option is obviously to avoid going type-theoretic in the first place.

[13] For instance, Williamson, 2003 argues that theorists who deny the coherence of absolutely unrestricted quantification cannot properly express the claim that absolutely no electron moves faster than the speed of light. But it can likewise be argued that theorists who deny the coherence of quantification across types cannot properly express Unique Existence, as this would require expressing the claim that, one exception apart, absolutely nothing (regardless of type) is a semantic value of some given expression.

[14] More precisely, schematic generalities can formalize Π^1_1-statements but not Σ^1_1 or anything more complex.

But it is not unreasonable to hold that natural language and the refinements thereof that are used in mathematics have a type-theoretic structure. Even if this is granted, however, we still have the option of lifting all type-restrictions associated with our object-language when we describe its semantics in some meta-language. We do this by allowing the first-order variables of the meta-language to range over *all* semantic values assigned to expressions of the object-language, regardless of the expressions' type. I will refer to this move as a *nominalization*. (The phenomenon of nominalization is familiar from the syntax of natural language, where it is often permissible to convert an expression that isn't a singular term (for instance '. . . is red') into one that is (in this case, 'redness').)

But the suggestion that we carry out a thoroughgoing nominalization when developing the semantics of our object-language comes at a cost. The cost is the reinstatement of the first premise **Sem1** of the Semantic Argument; for the first-order variables of the meta-language will then be allowed to range over interpretations. This will remove the type-theorists' defense against paradox. To avoid paradox, by far the most natural move will then be to reject the second premise **Sem2**, which allows the formation of the Russell-entity. But this rejection must not be some *ad hoc* trick invented merely to avoid paradox. Ideally, the rejection should be based on restrictions that are natural given a proper understanding of the entities in question. This is obviously a tall order. In what follows I will attempt to find and defend such a natural way of rejecting **Sem2**. Although this undertaking is fraught with difficulties, it can hardly be denied that it deserves to be explored. For as we have seen, the main alternatives all involve some degree of Semantic Pessimism. And although we have no *a priori* guarantee that the kind of semantic explicitness that we desire is possible, this is no excuse for not exploring potentially attractive ways in which it may be achieved.

5 Sets and properties

Given that we have rejected the type-theoretic responses, we have no choice but to take the semantic value of a predicate to be an *object* of some sort. Moreover, given that we want to allow quantification over absolutely everything, we have no choice but to accept that a predicate can be true of absolutely everything. Such a predicate must thus have as its semantic value an object that somehow collects or represents absolutely all objects, including itself.

Existing attempts to allow such objects have, as far as I know, always suggested that we trade traditional ZFC set theory for some alternative set-theory that allows universal sets. Such suggestions have been very unpopular, and rightly so. Traditional ZFC set theory is an extremely successful theory, which rests on a powerful conception of what sets are, namely the iterative conception.[15] By contrast, all known set theories with a universal set, such as Quine's New Foundations,[16] are not only technically unappealing but have lacked any satisfactory intuitive model or conception of the entities in question. It would therefore be folly to trade traditional ZFC for one of these alternative set theories.

But recall that what is needed is an object that "somehow collects or represents absolutely all objects." *Why does this object have to be a set?* After all, this object is needed as the semantic value of a certain predicate, and predicates are more plausibly taken to stand for concepts or properties than to stand for sets. For a predicate is associated with a condition that an object may or may not satisfy, and such conditions are more like concepts or properties than like sets. For instance, such conditions can be negated. Since concepts and properties have complements whereas (ordinary) sets don't, this means that conditions are more like the former than the latter. In fact, since (ordinary) sets don't have complements, they are extremely poorly suited to serve as the semantic values of predicates. So rather than *replacing* standard ZFC set theory with a non-standard set theory with a universal set, perhaps it is better to *supplement* it with a theory of properties, which can then include a property that is absolutely universal? We can then take interpretations to be countable sets that map well-formed expressions to their semantic values, and we can insist that predicates be mapped to properties rather than to sets. The challenge confronting us is then to articulate a conception of properties which (just like that of sets) is independently appealing, and which justifies enough properties to serve the semantic needs that we have delineated.

I will now argue that sets and properties relate to objects in very different ways, each of which is independently legitimate and interesting.[17] A set relates to objects in a *combinatorial* way by combining or collecting many objects into one. Each object among the many objects that are collected into a set is said to be *an element* of this set. It is essential to a set that it has precisely the elements that it in fact has: This set could not be the object it is had it not had precisely these elements. A property relates to objects in an *intensional* way by specifying a universally defined condition that an object must satisfy in order to possess the property. It is not essential to this property that it applies to precisely those objects to which it in fact applies. Rather, it is essential to the property that it applies to all and only such

[15] It has been argued that the iterative conception of sets favors a set-theory somewhat different from ZFC. For instance, Boolos, 1971 argues that it favors Zermelo set-theory Z. But nothing in what follows turns on precisely ZFC being the theory that is most naturally motivated by the iterative conception.

[16] See Quine, 1953 and Forster, 1995.

[17] My distinction between sets and properties is inspired by a similar distinction that Charles Parsons draws between sets and classes. See especially Parsons, 1974 and Parsons, 1977.

objects as satisfy the condition associated with the property: This property could not have been the property it is had it not applied to all and only such objects.

My defense of these claims will make use of a theory of *individuation*. This theory will be stated in a way that is sufficiently abstract to be shared by most philosophers who believe in a theoretically interesting notion of individuation. To produce a fully satisfactory account of individuation, this account would obviously have to be fleshed out and defended. But since the present article can afford to remain neutral on these difficult issues, I will not attempt to do so here.[18]

According to our abstract theory, individuation is based on two elements. Firstly, for every sort of object that can be individuated, there is a class of *fundamental specifications* of such objects. To use Frege's classic example, directions are most fundamentally specified by means of lines or other directed items. Secondly, for every sort of object that can be individuated, there is an equivalence relation on the fundamental specifications of such objects which states when two such specifications determine the same object. Continuing with Frege's example, the equivalence relation associated with directions is that of parallelism: two lines or other directed items determine the same direction just in case they are parallel. I will refer to such equivalence relations as *unity relations*. I will say that a specification and a unity relation *individuates* the object that the former determines in accordance with the latter.

Let's apply this theory to sets. A set is most fundamentally specified by means of a plurality of objects. Such pluralities can be represented by plural variables (as in Section 4).[19] Let's write $\textsc{Form}(uu, x)$ for the claim that uu form a set x; that is, that there is a set x whose elements are precisely the objects uu. It is standardly thought that some pluralities are "too large" to form sets. But when two pluralities do form sets, the extensionality of sets requires that these sets be identical just in case the pluralities from which they are formed encompass precisely the same objects. I therefore claim that sets are individuated in accordance with the following principle:

(Id-Sets) $$\textsc{Form}(uu, x) \wedge \textsc{Form}(vv, y) \rightarrow [x = y \leftrightarrow \forall z(z \prec uu \leftrightarrow z \prec vv)]^{20}$$

[18]For one attempt to flesh out and defend the abstract theory of individuation described below, see my Linnebo, 200x.

[19]For an elegant use of plural logic to motivate the axioms of ZFC, see Burgess, 2004. Like Burgess, I allow "degenerate" pluralities consisting of only one or zero objects.

[20]In my Linnebo, 200y I defend the heretical view that *every* plurality forms a set. I avoid paradox by formulating and defending a restriction on the naive plural comprehension scheme (which says that for every condition, there are some things which are all and only the things that satisfy this condition). Let $\sigma(uu)$ be the set formed by uu. Sets are then individuated in accordance with the following, simpler principle: $\sigma(uu) = \sigma(vv) \leftrightarrow \forall z(z \prec uu \leftrightarrow z \prec vv)$.

Since sets are bona fide objects, they may themselves be among the objects that make up a plurality. This means that the process of forming sets can be iterated. Moreover, since any object can be among the objects that make up a plurality, we must modify standard ZFC set theory so as to allow urelements. Let *ZFCU* be like ZFC except that the axiom of Extensionality is restricted so as to apply only to sets.[21]

A property F is said to be *essential* to an object x just in case x must possess F in order to be the object it is.[22] The above account of how sets are individuated explains why it is essential to a set that it have the elements that it in fact has, and why any characteristics by means of which these elements are specified need not be essential to the set. To see this, let x be a set, and let uu be some objects that jointly form this set. Then an object y cannot be identical with x unless y is a set formed by some objects vv such that $\forall z(z \prec uu \leftrightarrow z \prec vv)$. This means that y cannot be identical with x unless y has precisely the same elements as x. But since vv can be specified by means of characteristics completely different from those used to specify uu, it need not be essential to x what characteristics its elements have.[23]

Turning now to properties, we begin by observing that it suffices for present purposes to consider *monadic* properties. For we will always be working in theories that allow the formation of sets, which means that an n-adic property can be represented as a monadic property of (set-theoretic) n-tuples. So officially I will henceforth only operate with monadic properties. But unofficially I will still often talk about polyadic properties or relations, with the understanding that these are represented in the way just described.

It is useful to approach properties by way of *concepts*, roughly in the sense of Frege (or of the conceptualist of Section 3). A concept is most fundamentally specified by means of some completely general condition. By 'condition' I mean any meaningful one-place predicate, possibly with parameters. An account of what it is for a condition to be completely general will be given in Section 7. Next, two completely general conditions $\phi(u)$ and $\psi(u)$ determine the same concept just in case they stand in some suitable equivalence relation, which I will write as $Eqv_u(\phi(u), \psi(u))$. For present purposes, all we need to assume about this equivalence relation is that it cannot be coarser than co-extensionality; that is, if $Eqv_u(\phi(u), \psi(u))$, then $\forall u(\phi(u) \leftrightarrow \psi(u))$. (Why this requirement suffices will become clear shortly.) Concepts are

[21]Unlike some axiomatization of ZFCU, ours will not have an axiom postulating the existence of a *set* of all urelements. The reason is that we want to allow there to be as many properties as there are sets.

[22]See Fine, 1994.

[23]However, if it is essential to one of x's elements that it possess some characteristic, it will be essential to x that it has an element with this characteristic. (Fine calls such properties *mediately* essential.)

then individuated as follows:

$$(\Lambda) \qquad \Lambda u.\phi(u) = \Lambda u.\psi(u) \leftrightarrow Eqv_u(\phi(u), \psi(u))$$

It is important to bear in mind that the Λ-terms are second-order terms.[24] An object x is said to *fall under* a concept $\Lambda u.\phi(u)$ just in case $\phi(x)$.

This account of how concepts are individuated shows their essential properties to be very different from those of sets. For the identity of a concept $\Lambda u.\phi(u)$ is essentially tied to its condition of application $\phi(u)$. Had there been other objects satisfying the condition $\phi(u)$ than there actually are, then these objects too would have fallen under the concept $\Lambda u.\phi(u)$. And had some of the objects that actually satisfy the condition $\phi(u)$ not done so, then they would not have fallen under this concept $\Lambda u.\phi(u)$. So the essential properties of a concept have to do with the condition that an object must satisfy in order to fall under the concept, not with those particular objects that happen to satisfy this condition. For instance, it is essential to the universal concept $\Lambda u(u = u)$ that absolutely every object fall under it.

Next, observe that basic logical operations such as negation, conjunction, and existential generalization preserve the complete generality of conditions to which they are applied; for instance, the negation of any completely general condition condition will in turn be a completely general condition. This means that the realm of concepts is closed under the algebraic counterparts of these logical operations. (These algebraic operations will be described in more detail in the next section.) This in turn means that concepts, unlike sets, are well suited to serve as the semantic values of predicates.

Let's now attempt to nominalize concepts, that is, to bring concepts into the range of the first-order variables. We must then replace the second-order Λ-terms with analogous first-order λ-terms. I will refer to the resulting entities as *properties*. I use this label solely because in English the nominalization of a concept is often called a property. For instance, the nominalization of the concept involved in 'Fido is a dog' is the property of being a dog. It is important to leave aside other philosophical connotations of the word 'property'. In what follows the word will be used exclusively in the sense of a nominalization of a concept.

[24]It may therefore be objected to (Λ) that identity is a relation that can only hold between objects, not between concepts. I have no quarrel with this view provided it be conceded that there is a (second-order) relation on (first-order) concepts which is *analogous to identity* in that it is an equivalence relation which supports the analogue of Leibniz's Law. I will henceforth ignore this complication and talk as if concepts can be identical.

Properties are then individuated by the following first-order counterpart of (Λ)

$$(\lambda) \qquad \lambda u.\phi(u) = \lambda u.\psi(u) \leftrightarrow Eqv_u(\phi(u), \psi(u))$$

with some suitable restriction on the conditions $\phi(u)$ and $\psi(u)$ to avoid paradox.

Next we introduce a predicate P to be true of all and only properties by laying down the following axiom scheme:

$$(P) \qquad P(\lambda u.\phi(u))$$

We also define the relation η of property possession by laying down:

$$(\eta) \qquad v \ \eta \ \lambda u.\phi(u) \leftrightarrow \phi(v)$$

Note that the predicates P and η aren't explicitly defined but that their meanings will depend on what properties there are. This will become important towards the end of this paper.

Because properties are just nominalized concepts, they inherit from concepts their essential properties, namely their conditions of application. This means that properties and sets have completely different essential properties, which in turn means that no property can be a set and that the two relations $\in$ and η are fundamentally different.

Since my concern in this paper is chiefly with pure mathematics and its semantics, a simplification is possible. In pure mathematics there appears to be no difference between truth and necessary truth. It is therefore customary to regard all mathematical concepts as extensional. I will therefore assume that the equivalence relation $Eqv_u(\phi(u), \psi(u))$ is simply that of co-extensionality. Although the ensuing properties will behave slightly oddly on objects whose existence is contingent, this is irrelevant to our present concerns.[25] With this simplification, properties are individuated as follows:

$$(\text{V}) \qquad \lambda u.\phi(u) = \lambda u.\psi(u) \leftrightarrow \forall u(\phi(u) \leftrightarrow \psi(u))$$

Since this is just Frege's famous Basic Law V, some restriction will have to be imposed on the conditions $\phi(u)$ and $\psi(u)$ so as to avoid paradox. The task of finding and defending some such restriction will be our concern in Section 7.

[25] My commitment to calling the resulting entities "properties" is not very deep anyway. Perhaps it would be better to call them "extensions." But this too is potentially confusing, as there is a long tradition of taking extensions to be sets.

It is often advantageous to adopt an axiomatization of our theory of properties different from the one based on (P), (η), and (V). We can do this by "factoring" (V) into two components, one representing its existential import, the other representing its criterion of identity for properties. The first component consists of a property comprehension scheme that specifies what properties the theory is committed to. That is, for every suitable condition $\phi(u)$ we include an axiom stating that it defines a property:

$$\text{(V}\exists\text{)} \qquad \exists x[Px \wedge \forall u(u \;\eta\; x \leftrightarrow \phi(u))]$$

The second component consists of the following axiom that provides a criterion of identity for properties:

$$\text{(V=)} \qquad Px \wedge Py \rightarrow [x = y \leftrightarrow \forall u(u \;\eta\; x \leftrightarrow u \;\eta\; y)]$$

This axiomatization uses the predicates P and η as primitives but does away with λ. We may still use the λ-notation, however, subject to the contextual definition that $\psi[\lambda u.\phi(u)]$ be understood as short for $\exists x[\forall u(u \;\eta\; x \leftrightarrow \phi(u)) \wedge \psi(x)]$.

6 A general semantics for first-order languages

I will now show how a theory of both sets and properties allows us to develop a general semantics for first-order languages. I will define the notion of an interpretation of a first-order language and outline some simple applications of it. As will become clear shortly, I will take an interpretation to be a set-theoretic function that maps well-formed expressions to their semantic values, some of which are properties. The resulting semantics illustrates how beautifully sets and properties can interact.

A *first-order language with identity* is a language $\mathcal{L}$ of the following form. The simple expressions of $\mathcal{L}$ divide into *logical constants*, *non-logical constants*, and *variables*. The logical constant are the connectives $\neg$, $\wedge$, and $\exists$, as well as the predicate $=$. These expressions have a fixed interpretation (as respectively negation, conjunction, existence, and identity). The non-logical constants of $\mathcal{L}$ divide into names a_i and n-ary predicates F_i^n (where $i < \omega$ and $0 < n < \omega$). Being non-logical, these expressions are assigned different values by different interpretations. Finally, $\mathcal{L}$ has first-order variables x_i for $i < \omega$. On the semantics to be developed, a variable is a mere place-holder and will as such not be assigned any semantic value on its own. A *singular term* is either a name or a variable. An *atomic formula* is an n-ary predicate applied to n singular terms. Something is a *formula* just in case it is either an atomic formula or can be obtained from formulas in the usual way by means of negation, conjunction, or existential generalization.

A *lexicon* for a first-order language $\mathcal{L}$ is a set-theoretic function that assigns to each of $\mathcal{L}$'s non-logical constants (but not to the variables) an appropriate semantic value: to each name an object, and to each predicate a property (which is just a special kind of object). An *interpretation of a language* $\mathcal{L}$ is a set-theoretic function that maps *any* well-formed expression of $\mathcal{L}$ to an appropriate semantic value in a way that respects the expression's logical structure (of which more shortly).[26] An interpretation of a language $\mathcal{L}$ *satisfies* an $\mathcal{L}$-theory T just in case all the axioms of T come out true under this interpretation.

Our first goal is to extend a given lexicon I^- to an interpretation I. The obvious way to proceed is by recursion on the formation rules of $\mathcal{L}$. We begin by interpreting an atomic formula ϕ of the form $\mathbf{P}(\mathbf{t_0}, \ldots, \mathbf{t_n})$. Assume the variables among the singular terms $\mathbf{t_i}$ are $x_{i_0}, \ldots, x_{i_k}$ for some $k \leq n$. Assume also that these variables are *naturally ordered*, in the sense that $i_p < i_{p+1}$ for each $p < k$. Where J is a lexicon or an interpretation, let $[\![E]\!]_J$ abbreviate $J(E)$; where there is no danger of confusion, the subscript will be dropped. For such J, we also define the following translation of terms into the meta-language (which we assume to have the same variables as the object-language $\mathcal{L}$):

$$\mathbf{t_i}^J = \begin{cases} \mathbf{t_i} & \text{if } \mathbf{t_i} \text{ is a variable} \\ [\![\mathbf{t_i}]\!]_J & \text{if } \mathbf{t_i} \text{ is a name} \end{cases}$$

The interpretation $[\![\phi]\!]_I$ of the atomic formula ϕ is then defined by the universal generalization of the following formula:

$$(I_{At}) \qquad \langle x_{i_0}, \ldots, x_{i_k} \rangle \ \eta \ [\![\phi]\!]_I \leftrightarrow \langle \mathbf{t_0}^{I^-}, \ldots, \mathbf{t_n}^{I^-} \rangle \ \eta \ [\![\mathbf{P}]\!]_{I^-}$$

If on the other hand the singular terms $\mathbf{t_i}$ are *all names*, then $[\![\phi]\!]_I$ won't be a property but a truth-value. Assume the truth-values are represented by the numbers 1 and 0 (which may in turn be represented by sets). In this special case, the left-hand side of (I_{At}) must be replaced by '$[\![\phi]\!]_I = 1$'. To see that (I_{At}) and the special case just mentioned are reasonable definitions, it helps to consider some simple examples. For instance, according to (I_{At}) the semantic values of the atomic formulas $F_0^2(x_1, x_0)$ and $F_0^2(a_0, x_0)$ are given by the universal

[26] To keep things simple, I will not consider interpretations that restrict the ranges of the quantifiers to a property x. But this would be a straightforward modification.

closures of the following formulas:

$$\langle x_0, x_1 \rangle \; \eta \; [\![F_0^2(x_1, x_0)]\!] \leftrightarrow \langle x_1, x_0 \rangle \; \eta \; [\![F_0^2]\!]$$
$$x_0 \; \eta \; [\![F_0^2(a_0, x_0)]\!] \leftrightarrow \langle [\![a_0]\!], x_0 \rangle \; \eta \; [\![F_0^2]\!]$$

This is as one would expect: $[\![F_0^2(x_1, x_0)]\!]$ is the converse of $[\![F_0^2(x_0, x_1)]\!]$ $(= [\![F_0^2]\!])$, and $[\![F_0^2(a_0, x_0)]\!]$ is the property obtained by assigning $[\![a_0]\!]$ to the first argument place of the relation $[\![F_0^2]\!]$.

To describe the recursion clauses for formulas in general, it is useful to let $\vec{x}$ abbreviate strings of variables of the form $\langle x_{i_0}, \ldots, x_{i_k} \rangle$. The semantic values of formulas in general are then given recursively by the universal closures of the following formulas

$$(I_\neg) \qquad \vec{x} \; \eta \; [\![\neg\phi]\!] \leftrightarrow \neg(\vec{x} \; \eta \; [\![\phi]\!])$$

$$(I_\wedge) \qquad \vec{z} \; \eta \; [\![\phi \wedge \psi]\!] \leftrightarrow (\vec{x} \; \eta \; [\![\phi]\!]) \wedge (\vec{y} \; \eta \; [\![\psi]\!])$$

$$(I_\exists) \qquad \vec{x}_0 \; \eta \; [\![\exists \mathbf{v} \phi]\!] \leftrightarrow \exists \mathbf{v}(\vec{x} \; \eta \; [\![\phi]\!])$$

where $\vec{x}$ is the tuple of variables occurring in ϕ, $\vec{y}$ is the corresponding tuple for ψ, $\vec{z}$ is the tuple of variables occurring in the conjunction $\phi \wedge \psi$, and finally $\vec{x}_0$ is as $\vec{x}$ except with $\mathbf{v}$ removed. We must also add special clauses for the cases where the formulas are sentences, ensuring that these sentences are assigned the right truth-values.

We would now like to formalize semantic theorizing about the object-language $\mathcal{L}$. The following two formal languages will be particularly important in this undertaking.

Definition 1 (a) Let $\mathcal{L}_0$ be the language of ZFCU, that is, the first-order language with identity whose only non-logical constant is the set membership predicate $\in$.

(b) Let $\mathcal{L}_1$ be as $\mathcal{L}_0$ except for containing an additional non-logical predicate constant η for property possession. We will also use the λ-notation, subject to the contextual definition that $\psi[\lambda u.\phi(u)]$ be understood as short for $\exists x[\forall u(u \; \eta \; x \leftrightarrow \phi(u)) \wedge \psi(x)]$.

Given that our semantic theorizing involves both sets and properties, it is natural to base our meta-theory T on $\mathcal{L}_1$.

Our first task will be to convert the above implicit definition of an interpretation (relative to some given lexicon) into an explicit definition. Assume our meta-theory T contains enough set theory to handle n-tuples and to carry out set-theoretic recursion on syntax. Then, on the assumption that T contains enough axioms to ensure the existence of the requisite properties, we can simply use set-theoretic recursion on syntax to provide an explicit definition of the interpretation I. This is an example of how nicely sets and properties interact. Note in particular that this construction would have been impossible had concepts not been nominalized to properties. For the elements of a set are always in the range of *first-order* variables.

Precisely what axioms concerning properties must T contain? In general terms the answer is that T must contain enough axioms to ensure the existence of all the properties invoked in the above recursion clauses. But let's be more specific. First we need the existence of a property i to interpret the identity predicate, that is, a property had by all and only ordered pairs of the form $\langle u, u \rangle$. Then we need a series of axioms that ensure that the realm of properties is closed under some simple algebraic operations corresponding to the logical operations that are used in the recursion clauses. These will be operations such as permuting the argument places of a property, evaluating a property of n-tuples at some particular object in one of its argument places (both of which are crucial for the clause governing atomic formulas), taking the complement of a property (which is crucial for negation), taking the intersection of two properties (which is crucial for conjunction), projecting a property of n-tuples onto all but one of its axes (which is crucial for the existential quantifiers), and taking the inverse of a projection (which is needed for the conjunction of two formulas with different variables). Let's call these *the basic operations.* The previous section explained why it is reasonable to assume that the realm of properties is closed under these operations. An appendix will provide a proper consistency proof.

Definition 2 (a) Let *the minimal theory of properties*, V^-, be the $\mathcal{L}_1$-theory whose non-logical axioms are (V=), a comprehension axiom ensuring the existence of the identity property i, and axioms ensuring that the basic operations are always defined on properties.

(b) Where X is some class of property comprehension axioms, let V^X be the $\mathcal{L}_1$-theory which in addition to the axioms of V^- also has the property comprehension axioms in X. In particular, let $\mathrm{V}^\in$ be the theory that extends V^- by a comprehension axiom ensuring the existence of a property ϵ had by all and only ordered pairs $\langle u, v \rangle$ such that $u \in v$.

(c) Let ZFCU_0 be the $\mathcal{L}_1$-theory that contains all the usual axioms of ZFCU except that the Replacement and Separation schemes are replaced by single axioms quantifying over properties. Separation is thus formalized as

$$\text{(Sep)} \qquad \forall x \forall p \exists y \forall u (u \in y \leftrightarrow u \in x \wedge u \; \eta \; p),$$

and likewise for Replacement.

(d) Let $\mathrm{ZFCU}_0 + \mathrm{V}^X$ be the $\mathcal{L}_1$-theory whose set-theoretic axioms are those in ZFCU_0 and whose property-theoretic axioms are those in V^X.

Say that a meta-theory T is *minimally adequate* if it contains the minimal theory of properties and enough set-theory to handle n-tuples and set-theoretic recursion on syntax. A great amount of general semantic theorizing can be formalized in any minimally adequate meta-theory. We have already seen how such theories allows us to extend any lexicon to an interpretation. This also ensures that such theories can do justice to the claims that eluded the type-theorists in Section 4. The first claim, *Infinity*, is no longer relevant, as we now have only one kind of semantic values, namely objects. *Unique Existence* says that an interpretation assigns a unique semantic value to each expression. This is established by an easy induction on syntax. *Compositionality* too can be stated and proved. For instance, one easily shows that the truth-value of a simple subject-predicate sentence depends only on the semantic values assigned to the subject and the predicate.

Moreover, the usual method of Gödelization enables us to code various syntactic and proof-theoretic properties in T. For instance, let $Int(I)$ and $Thm(x)$ formalize respectively the claim that I is an interpretation of $\mathcal{L}$ and the claim that x is the Gödel number of a logical theorem. Then T proves the formalizations of a number of semantic truths. The first fundamental result of this sort concerns how the interpretation of an $\mathcal{L}$-formula ϕ relates to a very natural translation of ϕ into the meta-language $\mathcal{L}_1$, which I will now describe. Given an $\mathcal{L}$-lexicon J (or an $\mathcal{L}$-interpretation J which is based on some lexicon) I have already defined a translation $t \mapsto t^J$ of singular terms of $\mathcal{L}$ into $\mathcal{L}_1$. This translation can be extended to $\mathcal{L}$-formulas as follows. Where ϕ is an atomic formula $\mathbf{P}(\mathbf{t_0}, \ldots, \mathbf{t_n})$, let ϕ^J be $\langle \mathbf{t_0}^J, \ldots, \mathbf{t_n}^J \rangle \ \eta \ [\![\mathbf{P}]\!]_J$. Let the translation commute with the logical connectives. Then we get the following lemma, which will be very useful in semantic theorizing about $\mathcal{L}$.

Lemma 1 (Interpretation Lemma) Let T be a minimally adequate meta-theory. If the $\mathcal{L}$-formula ϕ has free variables, let $\vec{x}$ be based on its free variables taken in their natural order. Then

$$\vdash_T Int(I) \rightarrow \forall \vec{x}(\vec{x} \ \eta \ [\![\phi]\!]_I \leftrightarrow \phi^I).$$

If on the other hand ϕ is a sentence, the consequent of the above conditional must be replaced by $[\![\phi]\!]_I = 1 \leftrightarrow \phi^I$. This special case becomes clearer if we introduce a predicate $Tr_I(x)$ defined as $[\![x]\!]_I = 1$. Then T proves that, whenever I is an interpretation, $Tr_I(x)$ expresses the property of truth on the interpretation I.

Proof. By induction in T on the complexity of ϕ.

Theorem 1 (Soundness Theorem) Let T be a minimally adequate meta-theory. Then T proves the soundness of first-order logic with respect to our semantics. More precisely, if we let $C(x)$ be a function that maps the Gödel number of a formula to the Gödel number of its universal closure, then

$$\vdash_T Thm(x) \wedge Int(I) \rightarrow Tr_I(C(x)).$$

Proof. Assume a Frege-Hilbert style axiomatization of first-order logic with identity. Since the translation $\phi \mapsto \phi^I$ commutes with the logical constants, it maps each axiom of the object theory to an axiom of the meta-theory T. By the Interpretation Lemma, T proves $Tr_I(C(x))$ for each axiom x and each interpretation I. It remains to show that satisfaction of $Tr_I(C(x))$ is preserved under applications of the inference rules. The Interpretation Lemma allows us to translate this question to the question whether the corresponding inferences are truth-preserving in the meta-language, which they clearly are. Hence the corollary follows by induction in T. □

We can in a similar way define the usual notion of logical consequence and prove basic truths about it.

Our definitions and results thus far have only involved universal generalizations over interpretations but no claims about the existence of specific interpretations. We have therefore been able to make do with a minimally adequate meta-theory. But claims about the existence of specific interpretations will be needed in order to develop the intended semantics of a given theory and to give a formal proof of its consistency. For the former purpose, we need a meta-theory that proves the existence of an intended interpretation. For the latter purpose, it is extremely useful to be able to prove the existence of an interpretation that satisfies the object theory. For these purposes we typically need more than a minimally adequate meta-theory. The following theorem provides an example of such reasoning.

Theorem 2 (General Semantics of ZFCU) In $\text{ZFCU}_0 + \text{V}^{\in}$ we can prove the existence of the intended interpretation of ZFCU. This means in particular that $\text{ZFCU}_0 + \text{V}^{\in}$ proves the consistency of ZFCU.

Proof. The only non-logical constant of the language $\mathcal{L}_0$ of ZFCU is the membership predicate '$\in$'. In $\text{ZFCU}_0 + \text{V}^{\in}$ we can define the intended lexicon I^- for $\mathcal{L}_0$ that assigns to '$\in$' the membership property ϵ. We can also define an interpretation I based on this lexicon. By the Interpretation Lemma, $\text{ZFCU}_0 + \text{V}^{\in}$ proves $Tr_I(\phi) \leftrightarrow \phi^I$ for every $\mathcal{L}_0$-sentence ϕ. But since I is the intended interpretation, ϕ^I can be simplified. Recall that $(u \in v)^I$ is $\langle u, v \rangle \ \eta \ [\![\in]\!]_I$, and that $[\![\in]\!]_I = \epsilon$. From the definition of ϵ it thus follows that $(u \in v)^I \leftrightarrow u \in v$. Since all of this is provable in $\text{ZFCU}_0 + \text{V}^{\in}$, this theory proves $Tr_I(\phi) \leftrightarrow \phi$ for every $\mathcal{L}_0$-sentence ϕ.

Next, observe that $\text{ZFCU}_0 + \text{V}^{\in}$ licences property comprehension on any $\mathcal{L}_0$-formula. Such property comprehension enables us to derive all the comprehension and separation axioms of ZFCU. Since $\text{ZFCU}_0 + \text{V}^{\in}$ thus proves every axiom of ZFCU, it follows from (a formalization of) the result of the previous paragraph that $\text{ZFCU}_0 + \text{V}^{\in}$ also proves that every axiom of ZFCU is true on the intended interpretation. By the Soundness Theorem, $\text{ZFCU}_0 + \text{V}^{\in}$ proves that every theorem of ZFCU is true on the intended interpretation. But if ZFCU was inconsistent, not all of its theorems could be true on one and the same interpretation. □

A natural question at this point is whether we can go on and prove the existence of an interpretation that satisfies what we just used as our meta-theory, namely $\text{ZFCU}_0 + \text{V}^{\in}$. This question is answered affirmatively in Section 8, which describes an infinite sequence of theories of sets and properties, each strong enough to prove the existence of an interpretation satisfying all of the preceding theories.

Another natural question is what property comprehension axioms are true, or can at least consistently be added to ZFC set theory. The next two sections address the issue of truth, relying on a mixture of philosophical and mathematical considerations. An appendix gives a mathematical proof of consistency relative to ZFC plus the existence of an inaccessible cardinal.

7 What properties are there?

To be acceptable, a proposed restriction on the property comprehension scheme must satisfy two potentially conflicting requirements. Firstly, the restriction must be liberal enough to allow the properties we need in order to carry out the desired kind of semantic theorizing. Secondly, the restriction must be well motivated. As an absolute minimum, the restriction must give rise to a consistent theory. But ideally, the restriction should be a natural one, given an adequate understanding of properties. The restriction should be one it would have been natural to impose anyway, even disregarding the fact that paradox would otherwise ensue. I will now develop an account of properties that attempts to walk this fine line between admitting too few properties (such that the desired kind of semantic theorizing cannot be carried out) and admitting too many (such that contradiction ensues).[27]

My account of what properties there are is based on the requirement that individuation be well-founded. According to this requirement, the individuation of some range of entities can only presuppose such objects as have already been individuated. A requirement of this sort is *prima facie* very plausible. To individuate is to give an account of what the identity of some range of entities consists in. If this account is to be informative, it cannot presuppose the very entities in question, since doing so would amount to presupposing precisely what we are trying to explain. I will now investigate how such a well-foundedness requirement is best formulated and understood.[28]

The well-foundedness requirement applies to both kinds of elements on which individuation is based: specifications and unity relations. The requirement that specifications not involve or presuppose the object to be individuated is fairly straightforward. Sets provide a nice example. I argued in Section 5 that sets are specified by means of pluralities of objects. The well-foundedness requirement then says that no such plurality can contain the very set that it is supposed to specify. This means that no set can be an element of itself. More generally, it gives rise to the familiar set-theoretic axiom of Foundation. Concepts and properties provide another example. Here it is required that a condition that is supposed to specify some concept or property not contain parameters referring to the very concept or property that we are attempting to individuate.

It is somewhat harder to tell what the well-foundedness requirement amounts to in the case of unity relations. The presence of any parameters in the characterization of these relations causes no problems: this will be handled as just discussed. The problem comes from the fact that the characterization of a unity relation often makes use of quantifiers. It is natural to think that, in order to determine the truth-value of a quantified statement, we need to consider all the corresponding instances. And clearly each instance involves or presupposes the entity with respect to which it is an instance. So from this natural thought it follows that a unity relation presupposes all entities in the ranges of its quantifiers. When this analysis of the presuppositions of a quantified statement is plugged into the well-foundedness requirement, we get the familiar Vicious Circle Principle, according to which the concept or property defined by a condition $\phi(u)$ cannot itself belong to the totality over which the variable u is allowed to range. This result would be disastrous for our project of developing a

[27] It should be noted that nothing claimed in the paper so far commits me to the particular account of properties that follows. All that is required is an account of properties that satisfies the two requirements just described.

[28] I won't here attempt any systematic defense of the resulting requirement. This would require fleshing out the abstract and minimal account offered in Section 5. For some ideas towards a defense, see Linnebo, 200x.

semantics for languages with genuinely universal quantification. For if a concept or property can never belong to the totality on which it is defined, then this totality cannot be completely universal. It would therefore be impossible to define a concept or a property that is genuinely universal.

Fortunately, this analysis of the presuppositions of a quantified statement is excessively strict. For although natural, the thought that the truth-value of a quantified statement requires consideration of all of its instances is incorrect. The most extreme example of this is the statement that absolutely everything is self-identical. The truth of this statement can be determined without consideration of a single instance. This means that we need a better analysis of what objects a quantified statement presupposes. I will here focus on the presuppositions carried by the conditions that define concepts and properties. Such conditions must be capable of occurring within the scope of absolutely universal quantifiers. What objects do such conditions presuppose?

The answer I would like to propose distinguishes between two kinds of presuppositions carried by a condition: An entity can be presupposed either for its mere *existence* or for its *identity*, that is, for being the thing it is. For an example of the former, consider the two conditions $u = u$ and $u \neq u$. Whether these two conditions are co-extensive depends on whether there is anything in the domain on which they are defined. Now, the domain we are interested in is the absolutely universal one. Since this domain contains all sets, this sort of presupposition will not be a problem, as there will always be enough sets available. I will also assume that the universal domain contains as many objects not yet individuated as there are sets. This is a very plausible assumption. We may always go on to individuate new mathematical objects. This assumption will prove to be important below.

What is it for a condition to presuppose an entity for its identity? Since what a condition does is distinguish between objects—those of which it holds and those of which it doesn't—this notion of presupposition should be spelled out in terms of what distinctions the condition makes. Now, a condition can only presuppose an object if it is able to distinguish this object from other known objects. I therefore propose that we analyze what it is for a condition to presuppose only entities that are already individuated in terms of the condition's not distinguishing between entities not yet individuated. This proposal can be made mathematically precise as follows. Consider permutations π that fix all objects already individuated and that respect all relations already individuated in the sense that for each such relation R we have

$$\forall x_0 \ldots \forall x_n (Rx_0 \ldots x_n \rightarrow R\pi x_0 \ldots \pi x_n).$$

My proposal is then that a condition $\phi(u)$ presupposes only entities that are already individuated just in case $\phi(u)$ is invariant under all such permutations.

When a formula presupposes only entities that have already been individuated, there is no obvious philosophical reason why it should not define a property. For we have specified in a non-circular way what this property would be. Nor is there any mathematical reason why such a formula should not define a property. To see this, begin by observing that such a condition clearly defines a *concept*. Next I claim that any concept individuated in accordance with the well-foundedness requirement can be nominalized. For each such concept can be represented by means of one of the objects not yet individuated. Since these concepts don't distinguish between objects not yet individuated, it doesn't matter which representative we choose. Moreover, since we have assumed that there are as many objects not yet individuated as there are sets, there will be enough objects to represent all the concepts definable by conditions in any reasonable language. And we can carry out this process as many times as there are sets. I therefore conclude that the independently motivated well-foundedness requirement on the definition of concepts gives us nominalization of concepts for free.

Let's apply the above analysis of the well-foundedness requirement to the task of justifying such properties as were needed for the semantic theory developed in the previous section. Assume we have individuated all sets and that we want to go on and individuate concepts and properties. Say that a permutation π of the universe is $\in$-*preserving* just in case $\forall x \forall y(x \in \ y \rightarrow \pi x \in \pi y)$. By transfinite induction one easily proves that $\in$-preserving permutations leave pure sets untouched. There are also objects that are partially individuated, such as the singleton of an object u that is not yet individuated. But $\in$-preserving permutations preserve precisely the relation between this object u and its singleton. To fix everything already individuated, it is therefore sufficient to require that a permutation be $\in$-preserving. The well-foundedness requirement, as analyzed above, therefore demands that the condition used to define a concept or a property be invariant under $\in$-permutations.

The following lemma says that a large class of conditions satisfy this requirement.

Lemma 2 (Indiscernibility Lemma) Let $\phi(v_0, \ldots, v_n)$ be an $\mathcal{L}_0$-formula, possibly with parameters referring to pure sets. Let π be a $\in$-preserving permutation. Then $\langle x_0, \ldots, x_n \rangle$ satisfies ϕ just in case $\langle \pi x_0, \ldots, \pi x_n \rangle$ satisfies ϕ.

The proof is an easy induction on the logical complexity of ϕ. (Note that when π is $\in$-preserving, we have $\pi\langle x_0, \ldots, x_n \rangle = \langle \pi x_0, \ldots, \pi x_n \rangle$.) $\Box$

Say that x and y are *set-theoretically indiscernible* (in symbols: $x \approx y$) just in case there is a $\in$-preserving permutation π such that $\pi x = y$. This is easily seen to be an equivalence relation. Note that any two urelements are set-theoretically indiscernible. Say that a property x is *set-theoretic* just in case it doesn't distinguish between objects that are set-theoretically indiscernible (that is, just in case $u \ \eta \ x \wedge u \approx u' \rightarrow u' \ \eta \ x$). Set-theoretic properties make maximally coarse distinctions among non-sets: if such a property applies to one non-set, it applies to all. The Indiscernibility Lemma thus says that any property defined by a $\mathcal{L}_0$-formula with parameters referring to pure sets must be set-theoretic.

We can give a particularly nice explanation of how set-theoretic properties are individuated if we avail ourselves of second-order logic. The second-order theory I will use is von Neumann-Bernays-Gödel set theory with urelements (henceforth, *NBGU*).[29] The intended interpretation of this second-order language will have the first-order variables range over absolute everything. Thus, when I speak below about "the whole universe," I mean *the whole thing.* I use capital letters as second-order variables and refer to their values as *classes.*

I begin by characterizing what is required for a formula to be universally defined, not just on objects individuated thus far, but on all such as are yet to be individuated.

Definition 3 M is a *mock universe* for a formula ϕ iff M is indistinguishable from the whole universe from the point of view of ϕ. More precisely, M is a *mock universe* for a formula ϕ iff there is an equivalence relation $\approx$ on the whole universe such that ϕ cannot distinguish between objects that are thus equivalent and such that $\forall x \exists y (My \wedge x \approx y)$.

Definition 4 M is an *individuating domain* for a formula ϕ iff

(a) M is a mock universe for ϕ;

(b) M does not involve any property defined by ϕ;

(c) ϕ is defined on all of M.

Thus, to ensure that a formula is defined on the whole universe, including objects yet to be individuated, it suffices to show that it has an individuating domain.

Let $V[I]$ be the standard model of ZFCU based on a proper class I of as many urelements as there are sets. Then we have the following lemma.

Lemma 3 $V[I]$ is an individuating domain for $\mathcal{L}_0$-formulas with parameters referring to pure sets.

[29] NBGU is the (conservative) extension of ZFCU that allows second-order variables and quantifiers, contains a predicative second-order comprehension scheme, but allows no bound second-order variables in instances of the Separation or Replacement schemes.

Proof. By the Indiscernibility Lemma, such formulas cannot distinguish between objects that are set-theoretically indiscernible. Since every object is set-theoretically indiscernible from some object in $V[I]$, this is a mock universe for $\mathcal{L}_0$-formulas. Next, $V[I]$ can be described without in any way presupposing properties. ($V[I]$ can for instance be simulated in the standard model V of ZFC by letting some of the pure sets serve as urelements). Finally, all $\mathcal{L}_0$-formulas are defined on $V[I]$. □

Next I state a corollary that ensures that any property defined by a formula of the kind in question is individuated by its behavior on $V[I]$.

Corollary 1 Let $\phi(u)$ and $\psi(u)$ be two $\mathcal{L}_0$-formulas, possibly with parameters referring to pure sets. Then the two properties $\hat{u}.\phi(u)$ and $\hat{u}.\psi(u)$ defined by these formulas are identical iff $\forall u(\phi(u) \leftrightarrow \psi(u))$ holds in $V[I]$.

Proof. The only non-trivial direction is right-to-left. Assume $\hat{u}.\phi(u) \neq \hat{u}.\psi(u)$. Then by (V=) there is an object x on which $\phi(u)$ and $\psi(u)$ disagreed. But then, by the above lemma, these formulas also disagree on some object y in $V[I]$ such that $x \approx y$. □

This strategy can easily be adapted to show that other classes of formulas have individuating domains. We may for instance allow the formulas to contain parameters referring to other kinds of objects already individuated or to contain predicates that are true only of such objects. Let X be the class consisting of all referents of such parameters and all objects of the sort that these predicates can be true of. Consider the class of permutations that not only preserve the $\in$-relation except that also fix all the objects in X. Let $\approx$ be defined as above but in terms of this more restrictive class of permutations. We can then prove a modified Indiscernibility Lemma to the effect that the relevant formulas cannot distinguish between $\approx$-equivalent objects. We can also prove that, when I is a proper class of new urelements, $V[X \cup I]$ is an individuating domain for these formulas.

I now turn to the second of the two requirement on a theory of properties that were described at the beginning of this section. According to this requirement, the theory of properties must in a natural and well-motivated way rule out such properties as would give rise to paradox. Since the most obvious threat of paradox stems from the Semantic Argument, I will examine what this argument looks like in our present setting. Recall that we are taking interpretations to be countable sets that map expressions to objects. This means that the premise **Sem1**, which says that interpretations are objects, is incontrovertible. But we have also stipulated that interpretations are to map predicates to properties. Our hope is that this will provide a principled reason for rejecting the other premise **Sem2**, which permits the definition of a Russell-like property.

Assume the predicate letter 'P' is first in the series of non-logical expressions to which an interpretation must assign values. Then **Sem2** permits the following definition of a Russell-like property r_0:

$$(\text{Def-}r_0) \qquad \forall x(x\ \eta\ r_0 \leftrightarrow \neg\, x\ \eta\ x(0))$$

How plausible is it to reject this attempted definition? With some minimal assumptions, we can show that this definition is permissible just in case the following definition of a more traditional Russell property r is permissible:

$$(\text{Def-}r) \qquad \forall x(x\ \eta\ r \leftrightarrow \neg\, x\ \eta\ x)$$

Our question is thus how plausible it is to reject (Def-r).

I claim that we have good reason to reject this attempted definition. Let's begin by considering the predicate η in its intended sense, as true of two objects u and x just in case x is a property possessed by u. I claim that, if this intended sense can be made out at all, the well-foundedness requirement disallows the predicate η from defining any concept or property. For η will then represent the relation of property possession for all properties whatsoever. But two properties x and y are identical just in case they are borne η by precisely the same objects, that is, just in case $\forall u(u\ \eta\ x \leftrightarrow u\ \eta\ y)$. This means that η in its intended sense would distinguish every property from every other, thus maximally violating the well-foundedness constraint.

Although this gives us reason to be suspicious, the official verdict on the legitimacy of comprehension on formulas containing η must be based on the whatever meaning has *officially* been assigned to this predicate. The only such meaning comes from (V=) and the property comprehension axioms, which implicitly define η and the other primitive P (for being a property). I claim that already this official meaning makes η problematic from the point of view of the well-foundedness requirement. Assume we have individuated certain entities and now want to go on and individuate a property defined by '$u\ \eta\ x$'. The requirement is then that the condition '$u\ \eta\ x$' be invariant under all permutations that fix all objects and respect all relations individuated thus far. But this requirement isn't satisfied. For the second argument place of '$u\ \eta\ x$' distinguishes between objects yet to be individuated as non-properties (on which the condition will always be false) and objects yet to be individuated as non-empty properties (on which the condition will be true for suitable values of u).

Even so, we have not yet excluded the possibility that the Russell condition '$\neg\, u\ \eta\ u$' is less problematic than the more general condition '$u\ \eta\ x$'. To examine this possibility, assume we attempt to individuate the Russell property after having individuated certain other entities. The well-foundedness requirement then says that the Russell condition must be invariant under all permutations that fix all objects and respect all relations individuated thus far. But just as above, this requirement isn't satisfied, since the Russell condition distinguishes between objects yet to be individuated as non-properties (on which the condition will be true) and objects yet to be individuated as properties that possess themselves (on which the condition will be false). The well-foundedness requirement therefore disallows property and concept comprehension on the Russell condition as well.

8 A hierarchy of semantic theories

I have just argued that the well-foundedness requirement disallows property and concept comprehension on the condition '$u\ \eta\ x$'. But in order to avoid Semantic Pessimism, we will have to develop a general semantics for languages such as $\mathcal{L}_1$ that contain the predicate η. This means that some property will have to be assigned to this predicate as its semantic value. What could this semantic value be?

I begin with an informal description of my answer.[30] The problem we have encountered is that η—whether in its intuitive sense or its official sense fixed by the axioms—is concerned with objects yet to be individuated and therefore violates the well-foundedness requirement. But this problem can be avoided by adding to the condition '$u\ \eta\ x$' the restriction that x must be a property *already individuated*. The resulting condition does not distinguish between objects not yet individuated, and the well-foundedness requirement therefore allows it to define a property, which I will call e_0. When this new property e_0 is assigned to 'η' as its semantic value, we get an interpretation that satisfies $\text{ZFCU}_0 + \text{V}^{\in}$. Next, we can apply the basic operations to e_0 to define a whole range of new properties, which will thereby have been individuated. These new properties allow us to relax the restriction on the condition '$u\ \eta\ x$', now requiring only that x belong to the larger range of properties that have now been individuated. The resulting condition can again be shown to satisfy the well-foundedness

[30]In August 2005, two months after having completed this paper, I learnt from a talk by Kit Fine at the European Congress of Analytic Philosophy in Lisbon that he has carried out a construction very similar to the one I am about to describe; see his Fine, . Fine has also proved a number of technical results about his construction that go far beyond anything anticipated in this paper. He informs me that most of this work dates back to the academic year 1996-7; the construction is also alluded to in Fine, 1998, pp. 602 and 623.

requirement. It therefore defines a new property, which I will call e_1. When e_1 is assigned to 'η' as its semantic value, we get an interpretation that satisfies not only $\mathrm{ZFCU}_0 + \mathrm{V}^{\in}$ but also the property comprehension axiom by which e_0 was introduced. Continuing in this way, we get an infinite sequence of theories of sets and properties, each of which can be shown to be strong enough to prove the consistency of the preceding ones.

I now describe this construction in more detail. We begin by individuating some class of set-theoretic properties. For concreteness, assume we individuate those set-theoretic properties definable by formulas of $\mathcal{L}_0$ but allowing for parameters referring to pure sets. Now we want to use the set-theoretic properties we have just individuated to individuate more properties. We therefore look at permutations that not only respect the elementhood relation $\in$ but that fix the set-theoretic properties we have just individuated. By the argument of the previous section, any condition that is invariant under all such permutations defines a property. We would therefore like to characterize the conditions that are invariant in this way.

These conditions can clearly contain the predicate $\in$. But more interestingly, I will now show that they can also contain a two-place predicate ψ, which we get by adding to the condition '$u\ \eta\ x$' the restriction that x is one of the set-theoretic properties just individuated. I begin by observing that this predicate ψ is definable in $\mathrm{ZFCU}_0 + \mathrm{V}^{\in}$. For x is one of the set-theoretic properties just individuated if and only if x is definable from i and ϵ by basic operations (where any parameters must again refer to pure sets). We can therefore use set-theoretic recursion to define what it is for a property x to be definable from i and ϵ. This allows us to formulate the desired predicate $\psi(u, x)$. I next prove that $\psi(u, x)$ is invariant under the permutations just described. Let π be any such permutation. If x isn't one of the properties just individuated, then nor is πx. Hence it follows that both $\psi(u, x)$ and $\psi(\pi u, \pi x)$ are false. If, on the other hand, x is such a property, then $\pi x = x$. But since x is then a set-theoretic property, we have $\psi(u, x) \leftrightarrow \psi(\pi u, x)$. Combining these observations, we get $\psi(\pi u, \pi x) \leftrightarrow \psi(u, x)$, as desired.

Since $\psi(u, x)$ is invariant under the relevant class of permutations, it defines a property, which I will call e_0. We therefore adopt the following property comprehension axiom:

$$(\mathrm{V}^0) \qquad \exists e_0 \forall u \forall v [\langle u, v\rangle\ \eta\ e_0 \leftrightarrow \psi(u, v)]$$

I claim that, when e_0 is assigned to 'η' as its semantic value, we get an $\mathcal{L}_1$-interpretation that satisfies $\mathrm{ZFCU}_0 + \mathrm{V}^{\in}$. We begin by observing that since i and ϵ are definable from e_0, we can in $\mathrm{ZFCU}_0 + \mathrm{V}^0$ define an $\mathcal{L}_1$-lexicon that assigns to '$\in$' and 'η' respectively ϵ and e_0.

We have also seen how to extend this lexicon to an interpretation. This interpretation clearly satisfies ZFCU_0. Moreover, by our choice of ψ, this interpretation also satisfies the property-theoretic axioms in $\mathrm{V}^{\in}$.[31] This means that every axiom of $\mathrm{ZFCU}_0 + \mathrm{V}^{\in}$ is true under our interpretation. In fact, this claim itself is provable in $\mathrm{ZFCU}_0 + \mathrm{V}^0$, using the Interpretation Lemma. This means that $\mathrm{ZFCU}_0 + \mathrm{V}^0$ proves the consistency of $\mathrm{ZFCU}_0 + \mathrm{V}^{\in}$.

Our next task is to define an interpretation that satisfies our new theory $\mathrm{ZFCU}_0 + \mathrm{V}^0$. The minimal way of doing this is by introducing a property comprehension axiom (V^1) that ensures the existence of a property e_1 under which fall all and only ordered pairs $\langle u, x\rangle$ such that x is definable by basic operations from i, ϵ, and e_0 and such that u possesses this property x. As above, we can write down a comprehension axiom for e_1 and prove that this comprehension axiom satisfies the well-foundedness requirement. We can also prove that the resulting theory of sets and properties allows us to define an interpretation that satisfies $\mathrm{ZFCU}_0 + \mathrm{V}^0$.

I end by outlining how this process can be continued up through the ordinals. The process is carried out by set-theoretic recursion in a meta-theory containing ZFC. Assume the process has been carried out for all ordinals $\gamma < \alpha$. Then there is a (set-theoretic) sequence $\langle i, \epsilon, e_0, \ldots, e_\gamma, \ldots\rangle_{\gamma<\alpha}$ listing the properties explicitly licensed by comprehension axioms so far, each e_γ being defined as the property under which falls all and only ordered pairs $\langle u, x\rangle$ such that x is a property definable by basic operations from the preceding elements of the sequence and such that u possesses this property. Call such sequences *eta-sequences*. Let $EtaSeq(x)$ be a formalization in $\mathcal{L}_1$ of the claim that x is an eta-sequence. One easily proves that, given any two eta-sequences, they are either identical or one is an initial segment of the other. Let $Length(x, \alpha)$ be a formalization of the claim that x is a sequence of length α. Property comprehension axioms can then be stated as axioms asserting the existence of eta-sequences of various ordinal lengths:

(V^α) $$\exists x(EtaSeq(x) \wedge Length(x, \alpha + 2))$$

($\mathrm{V}^{<\infty}$) $$\forall \alpha(Ord(\alpha) \rightarrow \exists x(EtaSeq(x) \wedge Length(x, \alpha + 2)))$$

I now state a fundamental theorem about theories based on such comprehension axioms.

Theorem 3 (a) $\mathrm{ZFCU}_0 + \mathrm{V}^\alpha$ proves that, for any ordinal $\gamma < \alpha$, $\mathrm{ZFCU}_0 + \mathrm{V}^\gamma$ is consistent.

(b) $\mathrm{ZFCU}_0 + \mathrm{V}^{<\infty}$ proves that, for any ordinal α, $\mathrm{ZFCU}_0 + \mathrm{V}^\alpha$ is consistent.

[31] In fact, e_0 is the smallest property which, when interpreted as the relation of property application, satisfies the property-theoretic axioms in $\mathrm{V}^{\in}$.

Proof sketch. For (a), observe that $\text{ZFCU}_0 + \text{V}^\alpha$ proves the existence of an interpretation that maps 'η' to the (α+2)'th element of an eta-sequence, which we may refer to as e_α. This interpretation is easily seen to satisfy $\text{ZFCU}_0 + \text{V}^\gamma$ for each $\gamma < \alpha$. For (b), observe that $\text{ZFCU}_0 + \text{V}^{<\infty}$ proves that for each α there is an interpretation of the form just described. □

I show in an Appendix how to develop purely set-theoretic consistency proofs for all these theories of sets and properties.

9 Conclusion

I will end by taking stock of what has been accomplished. I began by explaining what I take to be the strongest argument against the coherence of unrestricted quantification, namely the Semantic Argument (Section 2). Then I outlined the type-theoretic response to this argument (Section 3). Although I believe this to be the strongest response developed to date, I argued that the type-theorists are unable to properly express certain important semantic generalizations about their view, and that in order to do better, we need a type-free theory where some sort of *object* can serve as the semantic values of predicates (Section 4).

Next I introduced a distinction between sets and properties, which I argued is natural and well motivated, and according to which properties are very well suited to serve as the semantic values of predicates (Section 5). I also showed how a theory of sets and properties can be used to develop a very explicit semantics for first-order theories whose quantifiers range over absolutely everything (Section 6).

The main challenge that remained at the end of Section 6 was to give an account of what properties there are. This account must satisfy two potentially conflicting requirements. On the one hand, the theory must allow enough properties to enable the desired sort of semantic theorizing. On the other hand, the theory must in a natural and well motivated way disallow such properties as would lead to paradox. I attempted to meet this challenge in Section 7 by means of the plausible requirement that individuation be well-founded. I showed that the resulting account allows all the properties needed for the semantic theory of Section 6, while simultaneously disallowing paradoxical properties such as the Russell-property postulated by the second premise **Sem2** of the Semantic Argument.

The most serious worry remaining at the end of Section 7 concerns the fact that the predicate η, in its intended sense, fails to define a property. Many people will no doubt feel this as a loss. But it should be kept in mind that the rejection of such a property is not an *ad hoc* trick to avoid paradox but follows from the independently motivated well-foundedness requirement. So perhaps we must give up as illusory our apparent grasp of an absolutely general relation of property application. What we do have, however, are relations of property application restricted to whatever properties have been individuated. And as we saw in Section 8, these restricted relations suffice to develop general semantics for all of our theories containing the predicate η.

No doubt, further investigations will be needed, especially of how unnatural it is to do without an absolutely general relation of property application. But I firmly believe that my arguments and results represent progress. I hope in particular to have shown that a first-order theory of sets and properties offers an approach to the problem of unrestricted quantification that is at least as promising as the popular type-theoretic responses of Section 3.[32]

Appendix. Set-theoretic consistency proofs

I will now describe a method for proving the consistency of various theories of sets and properties relative to ZFC plus the existence of an inaccessible cardinal. I will therefore only be concerned with *set-theoretic* models of our object theories. Although this means that second-order variables of the meta-language can be interpreted set-theoretically, I will continue to speak of the values of these variables as "concepts" and "relations." Recall also our convention from Section 7 of using lower- and upper-case letters as respectively first- and second-order constants and variables.

Definition 5 Let $\mathcal{M}$ be a model of ZFCU. Let E be a dyadic relation on its domain M. Let F be a concept on M. Say that E *nominalizes* a concept F iff there is an x such that $\forall u(Fu \leftrightarrow E\langle u, x\rangle)$. This x will then be said to *nominalize* F *under* E or to be *the nominalization of* F *under* E. Finally, say that E is *nominalizing* iff E nominalizes the identity relation and the concepts that E nominalizes are closed under the basic operations.

Note that for a model $\mathcal{M}$ of ZFCU to satisfy the minimal theory of properties V^-, there must be a nominalizing relation E on M that can interpret 'η'. Our goal will therefore be to construct models of ZFCU with suitable nominalizing relations. Unfortunately, our next lemma shows that we cannot have the sort of nominalizing relation that we most would have wanted.

[32] I am grateful to Agustín Rayo, Gabriel Uzquiano, Bruno Whittle, and Tim Williamson for valuable comments on earlier versions of this paper. Thanks also to audiences at the Universities of Bristol, Oslo and Oxford and at the Fifth European Congress for Analytic Philosophy in Lisbon, August 2005, for discussion.

Lemma 4 Let $\mathcal{M}$ be a model of ZFCU. Then no nominalizing relation E on M can nominalize itself.

Proof. Assume for *reductio* that some nominalizing relation E nominalizes itself. Then there is some x such that $E\langle u, v\rangle \leftrightarrow E\langle\langle u, v\rangle, x\rangle$. Since E is nominalizing, one of its nominalizations is the Russell property r, which is definable from x and the nominalization i of identity as $(\pi_0(x \cap i))^c$ by the basic operations of projection, intersection, and complementation. This definition of r ensures that $E\langle u, r\rangle \leftrightarrow \neg E\langle\langle u, u\rangle, x\rangle$. By substituting r for u we thus get $E\langle r, r\rangle \leftrightarrow \neg E\langle\langle r, r\rangle, x\rangle$. But from our choice of x we also have $E\langle r, r\rangle \leftrightarrow E\langle\langle r, r\rangle, x\rangle$. This produces a contradiction. □

This lemma leaves us with two main options. Either we can give up our insistence that all properties be total and assign to 'η' a partial property. Or we can maintain our view that all properties are total and instead make use of nominalizing relations that don't nominalize themselves. These relations will serve as lower approximations to the desired partial property. Although I have no principled objection to the former option, I will here pursue the latter.

Definition 6 Where κ is a cardinal, let $V^{<\kappa}[U]$ be the iterative hierarchy based on the class U of urelements but with the requirement that at every stage of the iterative construction of sets all sets be of cardinality $< \kappa$.

It is easily seen that, when κ is a regular cardinal, the construction of $V^{<\kappa}[U]$ terminates after κ steps. It is also easily seen that κ is inaccessible just in case $V^{<\kappa}[U] \models$ ZFCU.

Lemma 5 (Nominalization Lemma) Let κ be an infinite cardinal and U a collection of κ urelements. Assume there is a nominalizing relation E on $V^{<\kappa}[U]$. Assume there is a subclass $I \subseteq U$ containing κ urelements that don't nominalize any concepts under E. Let Q be a class of at most κ concepts on $V^{<\kappa}[U]$. Then it is possible to extend E to a nominalizing relation E^+ that nominalizes all the concepts in Q. Moreover, E^+ can be chosen such that there are still κ urelements that don't nominalize any concepts under E^+.

Proof. We begin by making some simplifying assumptions. We may assume that Q contains all the concepts nominalized by E. For upon addition of these concepts, the cardinality of Q will still be $\leq \kappa$. We may also assume that Q is closed under the basic operations. To see this, note first that the cardinality of $V^{<\kappa}[U]$ is κ. Using this, a simple cardinality argument shows that the cardinality of the closure of Q is still $\leq \kappa$.

Next we identify each member of Q that is not already nominalized by E with a unique member of I. We can do this in such as a way that $J = I \setminus Q$ is of cardinality κ. Then we let

E^+ be such that $E^+\langle u, x\rangle$ iff either $E\langle u, x\rangle$ or x has been identified with some concept $F \in Q$ such that Fu. By our assumptions regarding Q it follows that E^+ is a nominalizing relation that nominalizes all the desired concepts. By our choice of J, there are still κ urelements that don't nominalize any concepts under E^+. □

Theorem 4 ZFCU$_0$ + V$^\in$ is consistent.

Proof. Let κ be an inaccessible cardinal. Then $V^{<\kappa}[U]$ is a model of ZFCU. Let E be the empty relation. Use the Nominalization Lemma to produce an extended nominalizing relation E^+ that nominalizes the concepts defined by $u = v$ and $u \in v$. Interpreting 'η' as E^+ produces a model of ZFCU$_0$ + V$^\in$. □

Theorem 5 ZFCU$_0$ + V$^{<\infty}$ is consistent.

Proof sketch. Let $\mathcal{M}$ be the model of ZFCU$_0$ + V$^\in$ resulting from the proof of Theorem 4. Our task is to construct, for every ordinal $\alpha < \kappa$, an eta-sequence of length α. We proceed by induction on α. Let E_0 be the nominalizing relation on M emerging from the proof of Theorem 4 (where this relation was called E^+). Recall that E_0 can be chosen such that there is a collection I of κ urelements of M that don't nominalize concepts under E_0. So by the Nominalization Lemma, there is a larger nominalizing relation E_1 that nominalizes E_0 while still preserving a collection of κ urelements that don't nominalize concepts. Let e_0 be the nominalization of E_0 under E_1. For $\alpha = \beta + 1$, let E_β be the nominalizing relation that emerges from stage β. We then proceed as above to define an extended nominalizing relation E_α that nominalizes E_β, say as e_β. For α a limit ordinal, we let the new nominalizing relation E_α be the union of the nominalizing relations at the preceding stages. Our construction can be carried out such that at every step there are still κ urelements that don't nominalize any concepts under this new nominalizing relation. □

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